A New Approach in Nonparametric Estimation of Returns in Mean-Downside Risk Portfolio frontier

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Résumé. Depuis Markowitz (1952), l'analyse mathématique sur la gestion du portefeuille s'est développée considérablement, et la variance est devenue la définition mathématique la plus populaire du risque pour la sélection de portefeuille. Les chercheurs ont développé une variété de modèles en utilisant la variance comme mesure du risque dans diverses situations. En revanche, quand les distributions des rendements sont asymétriques, la sélection du portefeuille basée sur la variance peut être un vrai handicap. Pour palier à ce problème, la semi-variance a été proposée comme une autre mesure alternative du risque. C'est une mesure du Downside Risk. Dans le modèle d'optimisation de portefeuille développé dans ce cadre, les investisseurs minimisent uniquement les rendements inférieurs au taux de rendement *benchmark* (cible) tout en gardant les rendement désirables au dessus de cette cible. Le modèle moyenne-downside risk utilise une matrice dite *matrice de semivariance-covariance* . Cette matrice est endogène aux poids des actifs constituants le portefeuille ce qui rend le problème d'optimisation difficile à résoudre. Athayde (2001) explicite un algorithme itératif convergeant (utilisant la méthode de Lagrange à chaque itération) pour résoudre le problème d'optimisation. Néanmoins, il souligne que, pour un nombre limité d'observations, la frontière efficiente présente une discontinuité. Pour contrecarrer cette faiblesse, il propose dans un article de (2003) de remplacer les données par des moyennes pondérées calculées par la méthode du noyau. Ben Salah et al (2014) proposent de remplacer son estimateur par un autre plus robuste basé sur l'estimation non paramétrique de la médiane conditionnelle. Bien que donnant des résultats meilleurs que les précédentes, cette méthode est lente à converger. Cette lenteur est due à la ré-estimation, à chaque itération, des rendements global du portefeuille et ceux de toutes les actions à chaque instant. Dans cette communication, nous proposons une amélioration sensible de cet algorithme en commençant par remplacer tous les rendements des actions par leurs estimateurs non paramétriques (utilisant la moyenne ou la médiane), puis déduire les estimateurs des rendements du portefeuille et appliquer l'algorithme classique d'Athayde sur les données estimées. L'application de cette nouvelle approche sera effectuée sur un certain nombre de marchés nationaux et internationaux.

Mots-clés. Downside Risk, estimation non paramétrique, moyenne, médiane, méthode du noyau, semivariance.

Abstract. The Downside Risk (DSR) model for portfolio optimisation allows to overcome the drawbacks of the classical mean-variance model concerning the asymetry of returns and the risk perception of investors. This model optimization deals with a positive definite matrix that is endogenous with respect to portfolio weights. This aspect makes the problem far more difficult to handle. For this purpose, Athayde (2001) developped a new recurcive minimization procedure that ensures the convergence to the solution. However, when a finite number of observations is available, the portfolio frontier presents some discontinuity and is not very smooth. In order to overcome that, Athayde (2003) proposed a mean kernel estimation of the returns, so as to create a smoother portfolio frontier. This technique provides an effect similar to the case in which we had continuous observations. Ben Salah et al (2014), taking advantage on the the robustness of the median, replaced the mean estimator in Athayde's model by a nonparametric median estimator of the returns. While this method has a convincing results, its convergence is extremly slowly. In this paper, we propose significant improvements to this method which limit the number of iterations. The new approach is a subtle adaptation of the classical Athayde' s algorithm including nonparametric estimation. We analyse the properties of the improved portfolio frontier and apply this new method on real examples.

Keywords. Downside Risk, kernel method, median, mean, nonparametric estimation, semi-variance.

1 Introduction

Markowitz (1952) used mean return 's, variances and covariances to derive an efficient frontier where every portfolio maximizes the expected return for a given variance (or minimizes variance for a given expected return). This is popularly called the mean-variance criterion. While modern portfolio theory of Markowitz has revolutionized the investment world, it has also received substantial criticism.

The main criticism to variance, used by Markowitz (1952) as measure of risk is, in essence, that it gives the same importance and the same weight to gains and losses, also the use of variance suppose that returns are normally distributed. That is why Markowitz(1959) argues for another more plausible measure of risk that he calls the Semi-Variance wich takes into consideration the assymetry and the risk perception of investors.

However academics and practitioners still using the Mean-Variance approach to optimise their portfolios because it is easier to compute and have well-known closed-form solutions and deals with a symetric and exogenous covariance matrix, whereas mean-semivariance optimal portfolios cannot be easily determined. This follows from the fact that, the semi-covariance matrix is endogenous and not symetric (see for example Estrada (2004). Athayde (2003) maked use of nonparametric techniques to estimate smooth continuous distribution of the portfolio in question. He proposed to replace all the returns by their mean kernel estimators counterpart, and from these estimations, he optimized their DSR, constructing with this, a new portfolio frontier. Ben Salah et al (2014) proposed a new approach based on median kernel estimation of the returns, and compared their method

to Athayde's one. One can see that results obtained by the median estimation method are better than those obtained by the mean estimation one (smaller DSR and smoother frontier). Nevertheless, the computing step takes a longtime due to the construction of the estimators: the asset estimation returns are derived from the estimation of the portfolio returns and they change at each computing step.
In this paper, we propose to start by estimating nonparametrically all the returns of each asset. The estimation of all the returns of the portfolio is obtained as a linear combination of kernel mean or median estimation of the different assets returns. This article is organized into 2 sections. The next section is devoted to kernel estimation of returns and Downside risk. In section 2, we give an overview on the Althayde transfomed algorithm to get the optimal portfolio and its frontier.

2 Nonparametric Mean-Downside Risk Model

We assume that n assets are available, denote by r_{jt} the return return of asset j on date t, $T = 1, \dots, T$. In practice, only a finite number of observations are available, and although we can always have an enormous number of data, the assumptions of asset returns being identically distributed during the whole period of the sample may not be a very realistiq assumption. The idea behind this chapter is to make use of nonparametric techniques to estimate continuous distributions of the portfolios, and from these estimations, optimize their DSR, constructing with this, a new portfolio frontier. The first meaning of nonparametric covers technics that do not rely on data belonging to any particular distribution. In particular, they may be applied in situations where less is known about the application in question.

2.1 Kernel Mean and Median Return estimation

Let $K(.)$ be a probability density function *(the kernel)* and $h(T)$ a positive sequence *(the bandwidth, the smoothing parameter)* which tends to 0 as T tends to infinity. It is chosen in order to penalise the distance between r_{it} and r_{ij}. Assets returns are replaced by the following estimators:

- mean estimator : $\hat{r}_{jt} = \frac{\sum_{i=1}^{T} r_{ji} K(\frac{r_{jt}-r_{ji}}{h})}{\sum_{l=1}^{T} K(\frac{r_t - r_l}{h})}$, $j = 1, \dots, n$ and $t = 1, \dots, T$,
- median estimator: $\hat{\hat{r}}_{jt} = arg \min_{z \in \mathbb{R}} \frac{\sum_{i=1}^{T} |r_{ji} - z| K(\frac{r_{jt}-r_{ji}}{h})}{\sum_{i=1}^{T} K(\frac{r_{jt}-r_{ji}}{h})}$, $j = 1, n$ and $t = 1, \dots, T$.

The portefolio return estimators are derived from those of kernel nonparametric assets ones:
Let w_j be the percentage weight of the jth asset in the portfolio, the portfolio return on each time t is given by

$$r_{pt} = w_1 r_{1t} + w_2 r_{2t} + \dots + w_n r_{nt}$$

and its estimation is given by:

$$\tilde{r}_{pt} = w_1\tilde{r}_{1t} + w_2\tilde{r}_{2t} + ... + w_n\tilde{r}_{nt} \text{ where } \tilde{r}_{jt} = \hat{r}_{jt} \text{ or } \hat{\hat{r}}_{jt}.$$

2.1.1 The Kernel DSR estimation

The DSR is defined as:

$$DSR = \frac{1}{T}\sum_{t=1}^{T}[min(r_{pt} - B, 0)]^2, \ B \text{ is any benchmark return} \tag{1}$$

To get a kernel DSR estimation, we replace in the previous formulae all the observations $r_{pt}, t = 1, \dots, , T$ by their estimators $\tilde{r}_{pt}$ and we get

$$\widetilde{DSR}(w) = \frac{1}{T}\sum_{t=1}^{T}\left[\min(\tilde{r}_{pt} - B, 0)\right]^2, \tilde{r}_{pt} = w_1\tilde{r}_{1t} + w_2\tilde{r}_{2t} + ... + w_n\tilde{r}_{nt} \tag{2}$$

2.1.2 The DSR model optimization

The DSR optimization problem is the following:

$$\text{Minimize}_{(w_1,w_2,+...+w_n)} \quad \frac{1}{T}\sum_{t=1}^{T}[min(\tilde{r}_{pt} - B, 0)]^2 \text{ s.t } \sum_{j=1}^{n} w_j\mu_j = E^* \text{ and } \sum_{j=1}^{n} w_j = 1 \tag{3}$$

3 Kernel DSR minimization algorithm

3.1 DSR minimization

Let us denote by $\tilde{R}_{jt}$ the excess return of asset j on date t. That is $\tilde{R}_{jt} = \tilde{r}_{jt} - B$ where B is the chosed benchmark. Let $\tilde{M}$ the matrix with coefficients $\Sigma_{ijB} = \frac{1}{T}\sum_{t=1}^{k}\tilde{R}_{it}\tilde{R}_{jt}$ where k is the period in which the portfolio underperforms the target return B.
The optimization of assets portfolio can be modeled as follows

$$\text{Minimize}_{(w_1,w_2,+...+w_n)} \quad w^t\tilde{M}w, \text{ s.t. } \sum_{j=1}^{n} w_j\mu_j = E^* \text{ and } \sum_{j=1}^{n} w_j = 1. \tag{4}$$

The optimal solution is obtained using the following iteratif algorithm where the lagrangian method is used in each step:

- **Step 1**. We start with an arbitrary portfolio $w^0 = (w_1^0, w_2^0, ..., w_m^0)$. For each date $t, t = 1, \dots, T$, the returns of this portofolio

$$r_{pt}^0 = w_1^0 r_{1t} + w_2^0 r_{2t} + ...w_m^0 r_{nt}$$

will be replaced by

$$\tilde{r}_{pt}^0 = w_1^0 \tilde{r}_{1t} + w_2^0 \tilde{r}_{2t} + ...w_m^0 \tilde{r}_{nt}. \tag{5}$$

From the previous estimators (24), we select the dates when the estimated returns of portfolio w^0 had negative excess returns. This set is called S_0. Let $\tilde{R}_{il}^0 = \tilde{r}_{il} - B$ and $\tilde{M}^{(0)}$ the following positive semidefinite matrix:

$$M^{(0)} = \frac{1}{T} \sum_{t \in S_0} \begin{bmatrix} (\tilde{R}_{1t})^2 & \tilde{R}_{1t}\tilde{R}_{2t} & \tilde{R}_{1t}\tilde{R}_{nt} \\ \tilde{R}_{2t}\tilde{R}_{1t} & (\tilde{R}_{2t})^2 & \tilde{R}_{2t}\tilde{R}_{nt} \\ \vdots & \vdots & \vdots \\ \tilde{R}_{nt}\tilde{R}_{1t} & \tilde{R}_{mt}\tilde{R}_{1t} & (\tilde{R}_{nt})^2 \end{bmatrix} \tag{6}$$

- **Step 2** is to find the portfolio w^1 that solves the following problem:

$$\text{Minimize}_{(w_1, w_2, + ... + w_n)} \quad w^t \tilde{M}^0 w \text{ s.t } \sum_{j=1}^{n} w_j \tilde{\mu}_j = E^* \text{ and } \sum_{j=1}^{n} w_j = 1.$$

Using *Lagrangian method*, the solution is the following

$$\begin{cases} w^1 = \frac{\alpha_0 E^* - \lambda_0}{\alpha_0 \theta - \lambda_0^2} \tilde{M}^{(0)^{(-1)}} \tilde{\mu} + \frac{\theta_0 - \lambda_0 E^*}{\alpha_0 \theta_0 - \lambda_0^2} \tilde{M}^{(0)^{(-1)}} \mathbf{1} \\ \text{where } \alpha_0 = \mathbf{1}^t \tilde{M}^{(0)^{(-1)}} \mathbf{1}, \lambda_0 = \tilde{\mu}^t \tilde{M}^{(0)^{(-1)}} \mathbf{1} \text{ et } \theta_0 = \tilde{\mu}^t \tilde{M}^{(0)^{(-1)}} \tilde{\mu}. \end{cases} \tag{7}$$

Step 3. Using (26), we calculate the new returns of the portfolio by the following:

$$\tilde{r}_{pl}^1 = w_1^1 \tilde{r}_{1t} + w_2^1 \tilde{r}_{2t} + ...w_m^1 \tilde{r}_{mt}, \tag{8}$$

and we construct $S_1 = \{l | 1 < l < T \text{ where } \tilde{r}_{pl}^1 - B < 0\}$.

Let $\tilde{R}_{il}^1 = \tilde{r}_{il}^1 - B, l = 1, \dots, T$ and $i = 1, \dots, n$. Using the above estimators, we built the followinfg new positive definite matrix $\tilde{M}^{(1)}$

$$\tilde{M}^{(1)} == \frac{1}{T} \sum_{t \in S_1} \begin{bmatrix} (\tilde{R}_{1t}^1)^2 & \tilde{R}_{1t}^1\tilde{R}_{2t}^1 & \tilde{R}_{1t}^1\tilde{R}_{mt}^1 \\ \tilde{R}_{2t}^1\tilde{R}_{1t}^1 & (\tilde{R}_{2t}^1)^2 & \tilde{R}_{2t}^1\tilde{R}_{mt}^1 \\ \vdots & \vdots & \vdots \\ \tilde{R}_{mt}^1\tilde{R}_{1t}^1 & \tilde{R}_{mt}^1\tilde{R}_{1t}^1 & (\tilde{R}_{mt}^1)^2 \end{bmatrix} \tag{9}$$

Using *Lagrangian method*, the solution is the following

$$\begin{cases} w^2 = \frac{\alpha_1 E^* - \lambda_1}{\alpha_1 \theta_1 - \lambda_1^2} \tilde{M}^{(1)^{(-1)}} \tilde{\mu} + \frac{\theta_1 - \lambda_1 E^*}{\alpha_1 \theta_1 - \lambda_1^2} \tilde{M}^{(1)^{(-1)}} \mathbf{1} \\ \text{where } \alpha_1 = \mathbf{1}^t \tilde{M}^{(1)^{(-1)}} \mathbf{1}, \lambda_1 = \tilde{\mu}^t \tilde{M}^{(1)^{(-1)}} \mathbf{1} \text{ et } \theta_1 = \tilde{\mu}^t \tilde{M}^{(1)^{(-1)}} \tilde{\mu}. \end{cases} \tag{10}$$

- **Step 4.** Again, find a new portfolio, select its negative observation, construct a matrix $\tilde{M}^{(2,)}$, repeat the minimization with $\tilde{M}^{(2)}$, find a new portfolio, select its negative deviations, and so on. The iterations will stop when the matrix $\tilde{M}^{(F+1)}$ will be the same (or nearly) as $\tilde{M}^{(F)}$. The minimum DSR portfolio with expected return E^* is given by:

$$\begin{cases} w^{F+1} = \frac{\alpha_F E^* - \lambda_F}{\alpha_F \theta_F - \lambda_F^2} \tilde{M}^{(F)^{(-1)}} \tilde{\mu} + \frac{\theta_F - \lambda_F E^*}{\alpha_F \theta_F - \lambda_F^2} \tilde{M}^{(F)^{(-1)}} \mathbf{1} \\ \text{where } \alpha_F = \mathbf{1}^t \tilde{M}^{(F)^{(-1)}} \mathbf{1}, \lambda_F = \tilde{\mu}^t \tilde{M}^{(F)^{(-1)}} \mathbf{1} \text{ et } \theta_F = \tilde{\mu}^t \tilde{M}^{(F)^{(-1)}} \tilde{\mu}. \end{cases} \tag{11}$$

3.1.1 The DSR frontier

In order to build the portfolio frontier, we will have to find the DSR for varyed expected return E^*. For a given E^*, the same procedure as in previous paraghs is used to determine the DSR of the the optimal portfolio. The problem is computationally less complicated than in Ben Salah (2014) since we have not new estimations of returns for every assets at each optimization step.
Application on real data, from different markets, are used to support all points of this paper. Initial results are encouraging and an R code is available from the authors upon request.

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