On periodic threshold $GARCH$ processes: Probabilistic structure and empirical evidence.

Abdelouahab Bibi and Ahmed Ghezal
Department of Mathematics, University of Constantine (1), Algeria
E-mails addresses: a.bibi@umc.edu.dz and a.ghezal@centre-univ-mila.dz

Résumé: Dans cette communication, nous proposons, une classe de processus $GARCH$ à seuil à coefficients périodiques ($PTGARCH$). Pour ces processus, nous donnons des conditions assurant la stationnarité (au sens périodique) stricte et faible, l'existence des moments et la représentation $ARMA$. Le concept d'ergodicité géometrique et de $\beta-$mélange des modèles $PTGARCH$ est aussi étudié. Une approche par quasi-maximum de vraisemblance est proposée pour estimer les paramètres du modèle et ses propriétés asymptotiques.

Abstract: In this talk, we propose a natural extension of threshold $GARCH$ ($TGARCH$) processes to periodically time-varying coefficients ($PTGARCH$). Some theoritical probabilistic properties of $PTGARCH$ are discussed. This models, can be viewed as a special of random coefficient $GARCH$ models. For this class of processes, firstly, we establish theoritical conditions, which ensure that the process in the threshold model is strictly and second$-$order stationary (in periodic sense). Secondary, we derive conditions ensuring the existence of moments of any order. As a consequence, we observe that some subclass have the $\mathbb{L}_2-$structures of threshold periodic $ARMA$ processes ($PTARMA$) and hence admit $PARMA$ representation. The concept of geometric ergodicity and $\beta-$mixing of $PTGARCH$ processes are also discussed under general and tractable assumptions. These results are applicable to standard $GARCH$ models and have statistical implications such as parameter estimation and order identification. Some examples as special cases are proposed and studied.

Keywords: Periodic Threshold $GARCH$ model, stationarity, higher order moments, geometric ergodicity and $\beta-$mixing.

1 Introduction

A process $\varepsilon=(\varepsilon_n)_{n\in\mathbb{Z}}$ defined on some probability space $(\Omega,\Im,P)$ is called a periodic $PTGARCH\,(p,q)$ process with periodic $s>0$ if it is solution to the following stochastic difference equation

$$\forall n\in\mathbb{Z}:\varepsilon_n=h_ne_n \text{ and } h_n=\alpha_0(s_n)+\sum_{i=1}^{q}\left(\alpha_i(s_n)\varepsilon_{n-i}^{+}+\beta_i(s_n)\varepsilon_{n-i}^{-}\right)+\sum_{j=1}^{p}\gamma_j\left(s_n\right)h_{n-j} \tag{1.1}$$

where $\varepsilon_n^{+}=\dfrac{|e_n|+e_n}{2}$, $\varepsilon_n^{-}=\dfrac{|e_n|-e_n}{2}$, $(s_n)_n$ is a periodic sequence of positive integers with finite state space $\mathbb{S}=\{1,...,s\}$ defined by $s_n:=\sum\limits_{k=1}^{s}k\mathbb{I}_{\Delta(k)}(n)$ with $\Delta(k):=\{sn+k,n\in\mathbb{Z}\}$ which refers to the stage or "season" of the periodic cycle at time n, $(e_n)_{n\in\mathbb{Z}}$ is a sequence of independent identically distributed ($i.i.d.$) random variables defined on the same probability space $(\Omega, \mathcal{A}, P)$ with zero mean and unit variance and e_k is independent of ϵ_n for $k > n$ (the independence of $(e_n)_{n\in\mathbb{Z}}$ may be relaxed to a martingale difference assumption).

In equation (1.1), the volatility process $(h_n)_{n\in\mathbb{Z}}$ depending at time n, not necessarily symmetrically, through the coefficients $\alpha_i(s_n)$ and $\beta_i(s_n)$ together on the modulus and the sign of the past innovations. By setting $n = st + v$, $\varepsilon_{st+v} = \varepsilon_t(v)$, $h_{st+v} = h_t(v)$ and $e_{st+v} = e_t(v)$, Model (1.1) may be equivalently written as

$$\forall t \in \mathbb{Z} : \varepsilon_t(v) = h_t(v)\, e_t(v) \text{ and } h_t(v) = \alpha_0(v) + \sum_{i=1}^{q} \big(\alpha_i(v)\varepsilon_t^+(v-i) + \beta_i(v)\varepsilon_t^-(v-i)\big) + \sum_{j=1}^{p} \gamma_j(v)\, h_t(v-j), \quad (1.2)$$

which we will make heavy use. In (1.2), $\alpha_0(v)$, $\alpha_i(v)$, $\beta_i(v)$ and $\gamma_j(v)$ with $i \in \{1, ..., q\}$ and $j \in \{1, ..., p\}$ are the model coefficients at season v and $\varepsilon_t(v)$ refers to ε_t during the $v - th$ "season" or regime, $v \in \{1, ..., s\}$ of cycle t. For the convenience, $\varepsilon_t(v) = \varepsilon_{t-1}(v+s)$, $h_t(v) = h_{t-1}(v+s)$ and $e_t(v) = e_{t-1}(v+s)$ if $v < 0$. The non-periodic notations (ε_t), (h_t), (e_t) etc. will be used interchangeably with the periodic notations $(\varepsilon_t(v))$, $(h_t(v))$, $(e_t(v))$ etc. Note that there are not restrictions to guarantee the positivity of $h_t(v)$. However, the parameters of $TGARCH(p,q)$ model have to be restricted to guarantee the stationarity (in periodic sense) and the existence of moment of some orders. The process $(\varepsilon_n)_{n\in\mathbb{Z}}$ is globally non stationary, but is stationary within each period, are becoming an appealing tool for investigating both volatility and distinct "seasonal" patterns with threshold effect and continue to gain a growing interest especially in finance and monetary economics. Before processing, some algebraic notations are used throughout the paper.

$I_{(n)}$ is the $n \times n$ identity matrix, $O_{(n,,m)}$ denotes the matrix of order $n \times m$ whose entries are zeros, for simplicity we set $O_{(n)} := O_{(n,n)}$ and $\underline{O}_{(n)} := O_{(n,1)}$. The spectral radius of squared matrix M is noted $\rho(M)$, $\|.\|$ refers to the standard norm in $\mathbb{R}^n$ or the uniform induced norm in the space $\mathcal{M}(n)$ of $n \times n$ matrices. $\otimes$ denotes the Kronecker product of matrices. $Vec(M)$ is the usual column stacking vector of the matrix M. For any $p > 0$, $\mathbb{L}_p = \mathbb{L}_p(\Omega, \Im, P)$ denotes the space of random variables X defined on some probability space $(\Omega, \Im, P)$ such that $E\{|X|^p\} < +\infty$. The usual norm in $\mathbb{L}_p$ is given by $\|X\|_p = E\{|X|^p\}$ if $p \in]0,1[$ and $(E\{|X|^p\})^{\frac{1}{p}}$ otherwise.

For non-periodic case, some probabilistic results have been established, in particular necessary and sufficient conditions of stationarity and ergodicity have studied by Gonçalves and Mends-Lopes [5]. The stationarity and ergodicity (in periodic case) of model (1.2) may be expressed as follows. Define the $p-$vectors $\underline{H} = (1, 0, ..., 0)'$, $\underline{\gamma}_{1:p}(v) := (\gamma_1(v), ..., \gamma_p(v))'$, $2q-$vector $\underline{\zeta}_{1:q}(v) := (\alpha_1(v), \beta_1(v), ..., \alpha_q(v),\ \beta_q(v))'$, $r = (2q+p)-$random vectors $\underline{\varepsilon}_t(v) := \big(\varepsilon_t^+(v), \varepsilon_t^-(v), ..., \varepsilon_t^+(v-q+1), \varepsilon_t^-(v-q+1), h_t(v), ..., h_t(v-p+1)\big)'$, $\underline{\eta}_v(e_t(v)) := \alpha_0(v)(e_t^+(v), e_t^-(v), \underline{O}'_{(2(q-1))}, \underline{H}')'$ and $r \times r-$ random matrix

$$\Gamma_v(e_t(v)) = \begin{pmatrix} \underline{\zeta}_{1:q-1}(v)\, e_t^+(v) & \alpha_q(v)\, e_t^+(v) & \beta_q(v)\, e_t^+(v) & \underline{\gamma}_{1:p-1}(v)\, e_t^+(v) & \gamma_p(v)\, e_t^+(v) \\ \underline{\zeta}_{1:q-1}(v)\, e_t^-(v) & \alpha_q(v)\, e_t^-(v) & \beta_q(v)\, e_t^-(v) & \underline{\gamma}_{1:p-1}(v)\, e_t^-(v) & \gamma_p(v)\, e_t^-(v) \\ I_{(2(q-1))} & \underline{O}_{(2(q-1))} & \underline{O}_{(2(q-1))} & O_{(2(q-1),p-1)} & \underline{O}_{(2(q-1))} \\ \underline{\zeta}_{1:q-1}(v) & \alpha_q(v) & \beta_q(v) & \underline{\gamma}_{1:p-1}(v) & \gamma_p(v) \\ O_{(p-1,2(q-1))} & \underline{O}_{(p-1)} & \underline{O}_{(p-1)} & I_{(p-1)} & \underline{O}_{(p-1)} \end{pmatrix}_{r\times r}. \quad (1.5)$$

With this notation, Equation (1.2) may be rewritten in Markovian state space,

$$\underline{\varepsilon}_t(v) = \Gamma_v(e_t(v))\underline{\varepsilon}_t(v-1) + \underline{\eta}_v(e_t(v)). \quad (1.6)$$

Since $(e_t)_{t\in\mathbb{Z}}$ is an $i.i.d$ process, the random matrices $(\Gamma_t(e_t))_{t\in\mathbb{Z}}$ are independent and periodically distributed $(i.p.d)$, so Eq (1.6) is the same as the defining equation for multivariate generalized periodic random coefficient autoregressive

$(RCAR)$ model, except that random matrix $\Gamma_t(e_t)$ is not independent of $\underline{\eta}_t(e_t)$ as is required in these models (see Aknouche and Guerbyenne [1] and the references therein).

2 Periodic stationarity

Periodically-varying coefficients models, are not only of interest in their own right, but, because of their connection with multivariate models with constant coefficients. This claim is based upon the following construction. Indeed, iterate equation (1.6) $s-$time to get the following equality

$$\underline{\varepsilon}_t(s) = \left\{\prod_{v=0}^{s-1}\Gamma_{s-v}(e_t\,(s-v))\right\}\underline{\varepsilon}_t(0) + \sum_{k=1}^{s}\left\{\prod_{v=0}^{s-k-1}\Gamma_{s-v}(e_t\,(s-v))\right\}\underline{\eta}_k(e_t\,(k)).$$

where, as usual, empty products are set equal to $I_{(r)}$. Set $\underline{\varepsilon}_t(0) = \underline{\varepsilon}_t$ (if there is no confusion) then

$$\underline{\varepsilon}_{t+1} = \Gamma(\underline{e}_t)\underline{\varepsilon}_t + \underline{\eta}\,(\underline{e}_t) \tag{2.1}$$

where $\underline{e}_t = (e_t(1),...,e_t(s))'\,,\Gamma(\underline{e}_t) = \left\{\prod_{v=0}^{s-1}\Gamma_{s-v}(e_t\,(s-v))\right\}$ and $\underline{\eta}(\underline{e}_t) = \sum_{k=1}^{s}\left\{\prod_{v=0}^{s-k-1}\Gamma_{s-v}(e_t\,(s-v))\right\}\underline{\eta}_k(e_t\,(k))$. It is worth noting that $(\Gamma(\underline{e}_t))_{t\in\mathbb{Z}}$ is a sequence of $i.i.d.$ random matrices independent of $\underline{\varepsilon}_k$ for $k \le t$, and $(\eta\,(\underline{e}_t))_{t\in\mathbb{Z}}$ is a sequence of $i.i.d.$ random vectors. So, it follows from Gladysev [8] that the $r-$dimensional equation (1.6) (or equivalently (1.2)) is strictly (resp. second-order, ergodic) periodically stationary (hereafter SPS) (resp. PC, PE) if and only if equation (2.1) is strictly (resp. second-order, ergodic) stationary. However, the key tool for studying the strict stationarity of equation similar to (2.1) is the top$-$Lyapunov exponent $\gamma_L\,(\Gamma)$ associated with the sequence of random matrices $(\Gamma(\underline{e}_t))_{t\in\mathbb{Z}}$ defined by

$$\gamma_L\,(\Gamma) := \inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1}\Gamma(\underline{e}_{t-j})\right\|\right\}\right\} \overset{\text{a.s.}}{=} \lim_{t\longrightarrow\infty}\left\{\frac{1}{t}\log\left\|\prod_{j=0}^{t-1}\Gamma(\underline{e}_{t-j})\right\|\right\} \tag{2.2}$$

in which the second equality can be justified using Kingman's [7] subadditive ergodic theorem and the existence of $\gamma_L\,(\Gamma)$ is guaranteed however by the fact that $E\left\{\log^+\|\Gamma(\underline{e}_t)\|\right\} \le E\left\{\|\Gamma(\underline{e}_t)\|\right\} < +\infty$, where $\log^+(x) = \max(\log x, 0)$ for any $x > 0$. This shows that $\gamma_L\,(\Gamma)$ is independent of the chosen norm, so $\gamma_L\,(\Gamma) = \lim_{t\longrightarrow\infty}\left\{\log\left\|\prod_{j=0}^{t-1}\Gamma(\underline{e}_{t-j})\right\|^{1/t}\right\} = \log\rho\left(\prod_{v=0}^{s-1}\Gamma_{s-v}\right)$ when $\Gamma(\underline{e}_t)$ is non-random matrix.

2.1 Strict periodic stationarity

Since $(e_t)_{t\in\mathbb{Z}}$ is a stationary and ergodic process, then $\left(\Gamma(\underline{e}_t), \underline{\eta}\,(\underline{e}_t)\right)_{t\in\mathbb{Z}}$ is also a strict stationary and ergodic process and $E\left\{\log^+\|\Gamma(\underline{e}_0)\|\right\} < \infty$ and $E\left\{\log^+\left\|\underline{\eta}\,(\underline{e}_0)\right\|\right\} < \infty$. Then we have the following theorem

Theorem 2.1 *Equation* (2.1) *has a strictly stationary solution given by*

$$\underline{\epsilon}_{t+1} = \sum_{k\ge 0}\left\{\prod_{j=0}^{k-1}\Gamma(\underline{e}_{t-j})\right\}\underline{\eta}\,(\underline{e}_{t-k}) \tag{2.3}$$

if and only if $\gamma_L(\Gamma) < 0$. Moreover, the series (2.3) converges absolutely almost surely and the solution process is unique, ergodic and causal.

Remark 2.1 *Even if the condition $\gamma_L(\Gamma) < 0$ could be used as a necessary and sufficient condition for the strict stationarity of equation similar to (2.1), it is of little use for practical checking of stationarity since this condition involve the limit of products of infinitely many random matrices. Hence, some simple sufficient conditions ensuring the negativity of $\gamma_L(\Gamma)$ can be given.*

1. *If* $E\left\{\log\left\|\left\{\prod_{v=0}^{s-1}\Gamma_{s-v}(e_t(s-v))\right\}\right\|\right\} < 0$ *or* $E\left\|\left\{\prod_{v=0}^{s-1}\Gamma_{s-v}(e_t(s-v))\right\}\right\| < 1$ *then* $\gamma_L(\Gamma) < 0$.
2. *If* $E\left\{\log\left\|\prod_{j=0}^{r}\Gamma(\underline{e}_{t-j})\right\|\right\} < 0$ *then* $\gamma_L(\Gamma) < 0$.
3. *If* $\rho(\Gamma) < 1$, *then* $\gamma_L(\Gamma) < 0$ *where* $\Gamma = E\left\{\prod_{v=0}^{s-1}\Gamma_{s-v}(e_t(s-v))\right\}$.

Corollary 2.1 *For $PTGARCH(1,1)$ the sufficient condition given in remark (2.1) reduces to* $E\left\{\log\left\{\prod_{v=1}^{s}|\delta_v(0)|\right\}\right\} < 0$ *where* $\delta_v(t) = \alpha_1(v)e_t^+(v-1) + \beta_1(v)e_t^-(v-1) + \gamma_1(v)$.

Remark 2.2 *A condition involving matrices of smaller dimension can be obtained. Indeed, assume that $\beta_i(v) = \pm\omega\alpha_i(v)$ for all $i \in \{1,...,q\}$ and $v \in \{1,...,s\}$ with $\omega \neq 1$,then with $\tilde{\underline{\varepsilon}}_t(v) := (\varepsilon_t^+(v) \pm \omega\varepsilon_t^-(v),...,\varepsilon_t^+(v-q+1) \pm \omega\varepsilon_t^-(v-q+1), h_t(v),...,h_t'(v-p+1))' \in \mathbb{R}^{r-q}$, we obtain the representation $\tilde{\underline{\varepsilon}}_t(v) = \tilde{\Gamma}_v(e_t(v))\tilde{\underline{\varepsilon}}_t(v-1) + \tilde{\underline{\eta}}_v(e_t(v))$with $\tilde{\underline{\eta}}_v(e_t(v)) = \alpha_0(v)\left(\omega(e_t(v)), \underline{O}'_{(q-1)}, \underline{H}'\right)'$ where $\omega(e_t) = e_t^+ \pm \omega e_t^-$ and hence a version similar to (2.1), $\tilde{\underline{\varepsilon}}_{t+1} = \tilde{\Gamma}(\underline{e}_t)\tilde{\underline{\varepsilon}}_t + \tilde{\underline{\eta}}(\underline{e}_t)$ in which $\left(\tilde{\Gamma}(\underline{e}_t)\right)$ are $(r-q)\times(r-q)-$random matrices with $\tilde{\underline{\varepsilon}}_t(0) = \tilde{\underline{\varepsilon}}_t$.*

2.2 Second-order periodic stationarity

In this subsection we examine the conditions ensuring the existence of a unique causal periodically ergodic and periodically correlated (PC) solution to (1.2). Formally a second-order process $(\varepsilon_t)_{t\in\mathbb{Z}}$ is said to be PC with period s, if for any integers t and k, $E\{\varepsilon_{t+s}\} = E\{\varepsilon_t\}$ and $Cov(\varepsilon_{t+s}, \varepsilon_{k+s}) = Cov(\varepsilon_t, \varepsilon_k)$, so when $s = 1$ a PC process is equivalent to second-order stationary process. PC time series are common in many scientific fields where the observed phenomena have significant periodic behavior in mean, variance and covariance structure, namely in meteorology, hydrology, finance and economy. For convenience, we shall considered the centred version of the state-space representation (1.6), i.e.,

$$\tilde{\underline{\varepsilon}}_{t+1} = \Gamma(\underline{e}_t)\tilde{\underline{\varepsilon}}_t + \tilde{\underline{\eta}}(\underline{e}_t) \tag{2.4}$$

The main properties of the representation (2.4) are summarized in the following proposition

Proposition 2.1 *Consider the $TGARCH(p,q)$ model (1.2) with state space representation (2.4), then the $\tilde{\underline{\varepsilon}}_l$ is orthogonal to $\tilde{\eta}(\underline{e}_k)$, i.e., $E\left\{\tilde{\underline{\varepsilon}}_l\tilde{\underline{\eta}}'(\underline{e}_k)\right\} = O$ for all $k \geq l$.*

Theorem 2.2 *Assume that $E\left\{e_t^4\right\} < \infty$ and*

$$\rho\left(E\left\{\Gamma^{\otimes 2}(\underline{e}_t)\right\}\right) < 1 \tag{2.6}$$

then the series $\underline{\tilde{\varepsilon}}_{t+1} = \sum_{k\geq 0}\left\{\prod_{j=0}^{k-1}\Gamma(\underline{e}_{t-j})\right\}\underline{\tilde{\eta}}\left(\underline{e}_{t-k}\right)$ *converges absolutely a.s and in mean and constitute the unique, causal, strictly stationary having moments up to second-order.*

By analogy with integrated $PGARCH$, processes, we define the integrated $PTGARCH$ $(IPTGARCH)$ as follows

Definition 2.1 *The* $PTGARCH$ *process* (1.2) *is called* $IPTGARCH$ *process if* $\rho(M) = 1$ *where* $M = E\left\{\Gamma^{\otimes 2}(\underline{e}_t)\right\}$.

Now, we consider the strict periodic stationarity of $IPTGARCH$ processes. For this purpose, we assume that the functions $\alpha_i(.)$, $\beta_i(.)$ and $\gamma_j(.)$ are positive for $i \in \{0, 1, ..., q\}$, $j \in \{1, ..., p\}$ (so h_t represents the conditional standard deviation of ε_t given $\Im_{t-1} = \sigma(\varepsilon_j, j \leq t-1)$), so according to the theorem on the irreducible non-negative matrices, there exists some $\underline{\vartheta} = (\vartheta_1, ..., \vartheta_r)' \in \mathbb{R}^r$ with positive entries and $\|\underline{\vartheta}\| = 1$ such that $\underline{\vartheta}' M = \underline{\vartheta}'$. Let $T = diag\{\vartheta_1, ..., \vartheta_r\}$, $\underline{\varepsilon}_t^* = T\underline{\varepsilon}_t$, $\Gamma^*(\underline{e}_t) = T\Gamma(\underline{e}_t)T^{-1}$, $\underline{\eta}^*(\underline{e}_t) = T\underline{\eta}(\underline{e}_t)$. Then by equation (2.1) we have

$$\underline{\varepsilon}_{t+1}^* = \Gamma^*(\underline{e}_t)\underline{\varepsilon}_t^* + \underline{\eta}^*(\underline{e}_t) \tag{2.7}$$

Lemma 2.1 *Assume that functions* $\alpha_i(.)$, $\beta_i(.)$ *and* $\gamma_j(.)$ *are positive for* $i \in \{0, 1, ..., q\}$, $j \in \{1, ..., p\}$ *and* $\rho(M) = 1$. *Then by choosing an appropriate matrix norm, the top-Lyapunov exponent* $\gamma_L(\Gamma^*)$ *associated with the sequence* $(\Gamma^*(\underline{e}_t))_{t\in\mathbb{Z}}$ *is strictly negative, so the equation* (2.7) *has a unique strictly stationary and ergodic solution. Moreover, the process solution is given by*

$$\underline{\varepsilon}_{t+1}^* = \sum_{k\geq 0}\left\{\prod_{j=0}^{k-1}\Gamma^*(\underline{e}_{t-j})\right\}\underline{\eta}^*\left(\underline{e}_{t-k}\right) \tag{2.8}$$

Proposition 2.2 *Under the conditions of lemma 2.1, the* $IPTGARCH$ *process defined by* (1.2) *has a unique* SPS *solution with infinite variance.*

Theorem 2.3 *Let* $(\underline{\tilde{\varepsilon}}_t)_t$ *be the stationary solution of model* (2.4). *Assume that* $E\left\{e_t^{2m}\right\} < \infty$ *for any* $m > 1$.

1. *If* $\rho\left(E\left\{\Gamma^{\otimes m}(\underline{e}_t)\right\}\right) < 1$ *then* $\underline{\tilde{\varepsilon}}_t \in \mathbb{L}_m$.
2. *Conversely, if* $\rho\left(E\left\{\Gamma^{\otimes m}(\underline{e}_t)\right\}\right) \geq 1$, *and if for any* $v \in \{1, ..., s\}$, $\alpha_0(v) > 0$, $\alpha_i(v) \geq 0$, $\beta_i(v) \geq 0$ *and* $\gamma_j(v) \geq 0$ *with* $i \in \{1, ..., q\}$ *and* $j \in \{1, ..., p\}$, *then there is no strictly stationary solution* $(\underline{\tilde{\varepsilon}}_t)_t$ *satisfying* (2.4) *such that* $\underline{\tilde{\varepsilon}}_t \in \mathbb{L}_m$.

The geometric ergodicity and the $\beta-$mixing of a Markovian time series has been investigated by several authors (see Bibi and Aknouche[2] for further detail). The basic tools, is the following assumption

***C*.0** $(\underline{e}_n)_n$ is i.i.d. and has a probability distribution function absolutely continuous with respect to the Lebesgue measure and such that the density function takes positive values $a.s.$ on its support $E \subset \mathbb{R}^s$.

Theorem 2.4 *Under* ***C*.0,** (2.2) *and if there are* $0 < \delta \leq 1$ *and* $\tau < 1$ *such that* $E\left\{\|\underline{e}_0\|^{\delta}\right\} \leq \tau$ *and* $\underline{e}_0 \in \mathbb{L}_\delta$, *then the process* $(\underline{\varepsilon}_n)_n$ *defined by* (2.1) *is geometrically ergodic. Moreover, if initialized from its invariant measure,* $(\underline{\varepsilon}_n)_n$ *is strictly stationary and* $\beta-$*mixing with exponential decay.*

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