A NONPARAMETRIC MODEL CHECKS FOR TIME SERIES WHEN THE RANDOM VECTORS ARE NONSTATIONARY AND ABSOLUTELY REGULAR

Echarif Elharfaoui [1] & Michel Harel [2]

[1] *LIMA, Département de Mathématiques, Faculté des Sciences, Université Chouaib Doukkali, route Ben-Maachou, B.P. 20, 24000 El Jadida Maroc, elharfaoui.charif@gmail.com*
[2] *IMT (UMR CNRS 5219), Université Paul Sabatier, 31062 Toulouse & ÉSPÉ de Limoges, 209 boulevard de Vanteaux, 87036 Limoges cedex France, michel.harel@unilim.fr*

Résumé. Dans cette note, nous étudions quelques méthodes générales pour tester un modèle paramétrique associé à une série chronologique markovienne à valeurs réelles lorsque les vecteurs aléatoires sont non stationnaires et absolument réguliers. Notre idée est d'utiliser un processus empirique marqué basé sur les résidus qui converge en loi vers un processus gaussien.

Mots-clés. Série chronologique markovienne, Processus empirique marqué, Fonction ψ-autorégressive, ψ-résiduelles, Moyenne conditionnelle, Norme de variation totale, Non stationnarité, Coefficient d'absolue régularité, Topologie de Skorohod, AR-ARCH général.

Abstract. In this Note, we study some general methods for testing the goodness-of-fit of a parametric model for a real-valued Markovian time series under nonstationarity and absolute regularity. For that, we define a marked empirical process based on residuals which converges in distribution to a Gaussian process with respect to the Skorohod topology. This method was first introduced by Koul and Stute (1999), and then widely developed by Ngatchou-Wandji (2002, 2008) [2-3] under more general conditions. Applications to general AR-ARCH models are given.

Keywords. Markovian time series, Marked empirical process, ψ-autoregressive function, ψ-residuals, Conditional mean function, Norm of total variation, Nonstationarity, Geometrical absolute regularity, Skorohod topology, General AR-ARCH model.

1 Introduction

The purpose of this Note is to study a general method for testing the goodness-of-fit of a parametric model for a Markovian time series. Now, we define our model.

Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of random variables with continuous functions F_i on $\mathbb{R}$. Assume that F_i admits a strictly positive density.

In this paper, we will suppose that the sequence $\{X_i\}_{i\in\mathbb{N}}$ is absolutely regular with the rate

$$\beta(n) = O(\tau^n), \quad 0 < \tau < 1. \tag{1}$$

Suppose that F_i converges to the distribution function F (for the norm of total variation noted $\|\cdot\|_{TV}$) which admits a strictly positive density. Put $F_{i,j}$ the distribution function of (X_i, X_j). Furthermore, assume that for any $l > 1$, there exists a continuous distribution function $\widetilde{F}_l$ on $\mathbb{R}^2$ admitting a strictly positive density such that

$$\|F_{i,j} - \widetilde{F}_{j-i}\|_{TV} = O(\rho_0^i), \quad 1 \le i < j \le n,\ n \ge 1,\ 0 < \rho_0 < 1 \tag{2}$$

for which there exists a sequence $\{\widetilde{X}_i\}_{i\in\mathbb{N}}$ of stationary random variables absolutely regular with rate (1) and $(\widetilde{X}_i, \widetilde{X}_j)$ has $\widetilde{F}_{j-i}$ as distribution function $(i < j+1)$.

Some literature is concerned with parametric modeling in that m is assumed to belong to a given family

$$\mathcal{M} = \{m(\cdot;\theta) :\ \theta \in \Theta\} \tag{3}$$

of function, where $\Theta \subset \mathbb{R}^p$ is a proper parameter set.

Consider the general hypothesis testing the null hypothesis $\mathcal{H}_0$ is a parametric regression model and belongs to a family given : $m \in \mathcal{M}$ versus the local alternatives $\mathcal{H}_{1,n} :\ m \equiv m_n \in \mathcal{M}_{1,n},\ n \ge 1$ or $\mathcal{M}_{1,n} = \{m = m(\cdot;\theta) + n^{-1/2} r :\ \theta \in \Theta\}$ where r is a function satisfying $E(r(\widetilde{X}_1)) \neq 0$. For this purpose, we consider an empirical process which, under $\mathcal{H}_0$, depends on the unknown but true parameter θ_0. We estimate this parameter by, say $\widetilde{\theta}_n$, and then plug this estimator in the expression of the empirical process. We next show that the resulting empirical process converges in the distribution to a noncentred Gaussian process which has the same limit covariance function.

It is assumed that under $\mathcal{H}_0$, $m_0(\mathbf{x}) = m(\mathbf{x}, \theta_0)$ for some true value parameter θ_0. The problem is how to estimate or to test for the hypothesis when θ_0 is unknown. A well known case is, of course, the linear model in which $m(\mathbf{x};\theta) = \mathbf{g}'(\mathbf{x})\theta$, $\mathbf{g}$ is a known vector-valued function. Now, to describe these procedures, let ψ be a nondecreasing real-valued function such that $\sup_i E|\psi(X_i - \nu)| < \infty$, for each $\nu \in \mathbb{R}$. Define the ψ-autoregressive function m_ψ by the requirement that

$$E\{\psi(X_i - m_\psi(\mathbf{X}_{i-1})) \mid \mathbf{X}_{i-1}\} = 0 \quad \text{a.s.} \tag{4}$$

Throughout we shall assume that the underlying process is ergodic, the nonstationary distribution F_i of the X_i's is continuous and that

$$\sup_i E\psi^2(X_i - m_\psi(\mathbf{X}_{i-1})) < \infty. \tag{5}$$

For that, we consider an empirical process such that under $\mathcal{H}_0$ this process depends of a parameter θ_0. First, we start by estimating the parameter and we prove that the empirical process converges in distribution to a certain centered Gaussian process when the parameter is replaced by its estimator $\widetilde{\theta}_n$. Under $\mathcal{H}_{1,n}$, the empirical process converges in distribution to a noncentered Gaussian process which has the same limit covariance function. Consider the nulle hypothesis $\mathcal{H}_0 : \; m_\psi(\cdot) = m(\cdot;\theta_0) \quad \text{for some } \theta_0 \in \Theta$.

Let $\widetilde{\theta}_n$ be a consistent estimator of θ_0 under $\mathcal{H}_0$ based on $\{X_i\}_{i\geq 0}$. Define

$$R^*_{n,\psi}(\mathbf{x}) = n^{-1/2}\sum_{i=1}^{n}\psi(X_i - m(\mathbf{X}_{i-1};\widetilde{\theta}_n))1\!\!1_{\{\mathbf{X}_{i-1}\leq \mathbf{x}\}}, \quad \mathbf{x}\in\mathbb{R}^d. \tag{6}$$

The process $R^*_{n,\psi}$ is a marked empirical process, where the marks, or the weights at $\mathbf{X}_{i-1}$, are now given by the ψ-residuals $\psi(X_i - m(\mathbf{X}_{i-1};\widetilde{\theta}_n))$. It is uniquely determined by the $\{\mathbf{X}_{i-1}\}$ and these residuals and vice versa. Tests for $\mathcal{H}_0$ can be based on an appropriately scaled discrepancy of this process. Let $\epsilon_i = X_i - m_\psi(\mathbf{X}_{i-1}),\; i\in\mathbb{N}$, be the innovations of our model which are a sequence of absolutely regular random variables satisfying (1). We are ready to state our first result. The main results will be to prove the weak convergence of the process $R^*_{n,\psi}$ with respect to the Skorohod topology under some reasonable conditions and to investigate the power of tests based on $R^*_{n,\psi}$.

2 Conditions and weak convergence of the marked empirical process

For simplicity, we now suppose $d=1$. We know that the process defined in (6) takes its values in the Skorohod space $D(-\infty,\infty)$ and the convergence in this space is equivalent to the weak convergence on compacts. This excludes the possibility of handling goodness-of-fit statistics such as $\sup_{x\in\mathbb{R}}|R^*_{n,\psi}(x)|$. To also deal with such statistics, we continuously extend $R^*_{n,\psi}$ to $-\infty$ and ∞ by setting : $R^*_{n,\psi}(-\infty) = 0$, $R^*_{n,\psi}(x)$ is defined by (6) for $x\in\mathbb{R}$ and $R^*_{n,\psi}(\infty) = n^{-1/2}\sum_{i=d}^{n}\psi(X_i - m(\mathbf{X}_{i-1};\widetilde{\theta}_n))$. Then $R^*_{n,\psi}$ becomes a process in $D[-\infty,\infty]$, which, modulo a continuous transformation, is the same as $D[0,1]$. Consider the sequence of distribution functions $\{\overline{F}_n\}_{n\geq 1}$ defined by $\overline{F}_n = n^{-1}\sum_{i=1}^{n}F_i$. For the behavior of the process $R^*_{n,\psi}$ defined in (6), some regularity assumptions on the estimator $\widetilde{\theta}_n$ will be needed. These conditions are similar to those of Koul and Stute [1] but our sequence X_i is nonstationary and geometrically absolutely regular, rather than being iid.

Condition 1. Under $\mathcal{H}_0$, that is $m = m(\cdot;\theta_0)$ for some unknown θ_0 in Θ, $\widetilde{\theta}_n$ admits an expansion : $n^{1/2}(\widetilde{\theta}_n - \theta_0) = n^{-1/2}\sum_{i=1}^{n}\mathbf{l}(X_i, X_{i-1};\theta_0) + o_p(1)$ for some vector-valued function $\mathbf{l}$ such that

(a) $\sup_i E\{\mathbf{l}(X_i, X_{i-1};\theta_0)|X_{i-1}\} = 0$ for any $i\geq 1$

(b) $L_{i,j} = E\{\mathbf{l}(X_i, X_{i-1};\theta_0)\mathbf{l}'(X_j, X_{j-1};\theta_0)\}$ exists for all $i,j\geq 1$.

Condition 2. (a) $m(x;\theta)$ is continuously differentiable at each θ in the interior set Θ^0 of Θ. Put

$$\mathbf{g}(x;\theta) = \frac{\partial m(x;\theta)}{\partial\theta} = \big(g_1(x;\theta),\ldots,g_p(x;\theta)\big)' \tag{7}$$

(b) there exists an $\{F_i\}_{i\geq 1}$ and F-integrable function $M(x)$ such that

$$\big|g_j(x;\theta)\big| \leq M(x), \quad \text{for all} \;\; \theta \in \Theta \;\; \text{and} \;\; 1 \leq j \leq p. \tag{8}$$

Condition 3. There exists a function m from $\mathbb{R}\times\Theta$ to $\mathbb{R}^q$ such that $m(.;\theta_0)$ is measurable and satisfies the following : for all $k < \infty$,

$$\sup_{1\leq i\leq n} \sup_{n^{1/2}\|t-\theta_0\|\leq k} n^{1/2}|m(X_{i-1};t) - m(X_{i-1};\theta_0) - (t-\theta_0)'m(X_{i-1};\theta_0)| = o_p(1) \tag{9}$$

and

$$\sup_i E\|m(X_{i-1};\theta_0)\|^{2+\delta} < \infty, \quad \text{for some} \;\; \delta > 0. \tag{10}$$

Theorem 1. *Under $\mathcal{H}_0$, assume that for any $u \in [0,1]$,*

$$\sup_{i\geq 1} E\big(|\psi(X_i - m(X_{i-1}))|^{2+\gamma_0} \mid U_{i-1} = u\big) < CE\big(|\psi(\widetilde{X}_1 - m(\widetilde{X}_0))|^{2+\gamma_0} \mid \widetilde{U} = u\big) < \infty, \; \gamma_0 > 0,$$

where $U_{i-1} = \overline{F}_n(X_{i-1})$, $1 \leq i \leq n$, $\widetilde{U} = F(\widetilde{X})$, C is some positive constant and the conditions (1) and (2) hold and let Conditions 1, 2 and 3 be satisfied, then $R^*_{n,\psi} \to R^*_{\infty,\psi}$ in distribution in the space $D[-\infty,\infty]$, where $R^*_{\infty,\psi}$ is a centred Gaussian process with covariance function

$$\begin{aligned} K^*_\psi(x,y) \;\; = \;\; & K_\psi(x,y) + \mathbf{G}'(x;\theta_0)\big(L_{1,1}(\theta_0) + 2\sum_{k=1}^{\infty} L_{1,k}(\theta_0)\big)\mathbf{G}(y;\theta_0) \\ & -\mathbf{G}'(x;\theta_0)\sum_{k=0}^{\infty} E\big(1\!\!1_{\{\widetilde{X}_0\leq x\}}\psi(\widetilde{X}_1 - m(\widetilde{X}_0;\theta_0))\mathbf{l}(\widetilde{X}_{k+1},\widetilde{X}_k;\theta_0)\big) \\ & -\mathbf{G}'(y;\theta_0)\sum_{k=0}^{\infty} E\big(1\!\!1_{\{\widetilde{X}_0\leq y\}}\psi(\widetilde{X}_1 - m(\widetilde{X}_0;\theta_0))\mathbf{l}(\widetilde{X}_{k+1},\widetilde{X}_k;\theta_0)\big), \end{aligned} \tag{11}$$

where $\mathbf{G}(x;\theta) = \big(\mathbf{G}_0(x;\theta),\ldots,\mathbf{G}_p(x;\theta)\big)'$, $\;\mathbf{G}_j(x;\theta) = \int_\infty^x g_j(u;\theta)dF(u)$, $\;0 \leq j \leq p$ and

$$K_\psi(x,y) = F(x\wedge y)Var(\psi(\widetilde{X}_1)) + 2\sum_{k=1}^{\infty} Cov(\psi(\widetilde{X}_1),\psi(\widetilde{X}_{1+k}))\widetilde{F}_k(x,y), \tag{12}$$

3 The testing procedure

From the results obtained in Theorem 1, some testing procedure can be derived. We can consider the Cramér-von Mises type test defined by

$$\mathcal{T}_n = \int (R^*_{n,\psi}(\mathbf{x}))^2\omega(\widehat{F}_n(\mathbf{x}))\mathrm{d}\widehat{F}_n(\mathbf{x}), \tag{13}$$

where ω is a weight function and $\widehat{F}_n$ is the empirical distribution function of the random vectors $\mathbf{X}_d, \ldots, \mathbf{X}_n$. We easily deduce that under the conditions of Theorem 1, $\mathcal{T}_n$ converges in law to $\mathcal{T} = \int (R^*_{\infty,\psi}(F^{-1}(u)))^2 \omega(u) \mathrm{d}(u)$. We remark that $\mathcal{T}_n$ can be also written as

$$\mathcal{T}_n = \frac{1}{n} \sum_{i=d}^{n} \omega(\widehat{F}_n(\mathbf{X}_{i-1})) \left\{ \sum_{j=d}^{n} \psi(X_j - m(\mathbf{X}_{j-1}; \widetilde{\theta}_n)) 1\!\!1_{\{\mathbf{X}_{j-1} \leq \mathbf{X}_{i-1}\}} \right\}^2.$$

The tails probability of the limiting distribution of the Cramer-von Mises test statistics would be very difficult to compute. That is why it is necessary to proceed to a discretization of $\mathcal{T}$ like in Ngatchou-Wandji [2], Ngatchou-Wandji and Harel [4].

As in Ngatchou-Wandji [2], the discretization that we can propose, follows from the Karhunen-Loève expansion of the processes $\mathcal{T}$. Denote by $W(\cdot) = R^*_{\infty,\psi}(F^{-1}(\cdot))$ the process defined on $[0,1]^d$. Its Karhunen-Loève expansion can be written as $W = \sum_{j=d}^{\infty} \lambda_j^{1/2} W_j f_j$, where $\lambda_d \geq \lambda_{d+1} \geq \ldots$ are the eigenvalues of the covariance operator $B_\psi(\cdot) = K^*_\psi\big(F^{-1}(\cdot), F^{-1}(\cdot)\big)$ which are supposed strictly positive, the sequence of functions $f_d, f_{d+1}, \ldots$ is a complete orthonormal base for $L^2[0,1]^d$ of eigenvectors of the operators of the operator B_ψ and the random variables $W_j = \lambda_j^{-1/2} \int_0^1 W(\mathbf{v}) f_j(\mathbf{v}) \mathrm{d}(\mathbf{v})$ are independent $\mathcal{N}(0,1)$ under $\mathcal{H}_0$.

Then it is possible to choose a test statistic on the form $\mathcal{T}_n^J = \sum_{j=d}^{J} W_{n,j}^2$, where $J > 1$ is the number of the more informative terms in the development (12) and for any $j \geq 1$

$$W_{n,j} = \lambda_j^{-1/2} n^{-1} \sum_{i=d}^{n} R^*_{n,\psi}(\mathbf{X}_{i-1}) \omega(\widehat{F}_n(\mathbf{X}_{i-1})) f_j(\widehat{F}_n(\mathbf{X}_{i-1})).$$

Under $\mathcal{H}_0$, $\mathcal{T}_n^J$ converges is law to $\mathcal{T}^J = \sum_{j=d}^{J} W_j^2$ which has asymptotically a chi-square distribution with J degrees of freedom. However, the λ_j's and f_j's are difficult to compute in practice. A way to overcome this difficulty was suggested by Ngatchou-Wandji [2,3] by approximating the integrals by discretization.

4 Applications to the AR-ARCH model

Now we apply the results of Section 3 to test an AR-ARCH model against an other AR-ARCH model. Consider a model which can be written in the form

$$X_i = m(X_{i-1}, \ldots, X_{i-d}; \theta) + v(X_{i-1}, \ldots, X_{i-d}) \epsilon_i, \quad i \geq 1 + d, \tag{14}$$

where $\theta \in \Theta \subset \mathbb{R}^p$ a proper parameter set, $m(\cdot)$ satisfying Conditions 2 and 3 and $v(\cdot)$ continuous are unknown.

Let $\{\mathbf{X}_{i-1}\}_{i \geq 1+d}$ denotes the random sequence of vectors in $\mathbb{R}^d$ defined by

$$\mathbf{X}_{i-1} = (X_{i-1}, \ldots, X_{i-d})', \quad i \geq 1 + d.$$

We suppose that the sequence $\{\mathbf{X}_{i-1}\}_{i\geq 1+d}$ satisfies the conditions (1) and (2) in the introduction and $\{\epsilon_i\}_{i\geq 1+d}$ is sequence of absolutely regular random variables satisfying (1). Formula (14) can be written as $X_i = m(\mathbf{X}_{i-1};\theta) + v(\mathbf{X}_{i-1})\epsilon_i, \quad i \geq 1+d$.
We will use the results of Section 3 to test :

$\mathcal{H}_0 : \ m(\cdot;\theta) \in \mathcal{M}$ versus the sequence of alternatives $\mathcal{H}_{1,n} : \ m(\cdot;\theta) \equiv m_n \in \mathcal{M}_{1,n}$.
We want to test the hypothesis $\mathcal{H}_0 : \ m(\cdot;\theta) \in \mathcal{M}$ against the local alternatives $\mathcal{H}_{1,n}$: $m(\cdot;\theta) = m(\cdot;\theta) + n^{-1/2}r(\cdot), \quad \theta \in \Theta$, where r is a function which has the same properties as v and $E(r(\widetilde{X}_1)) \neq 0$.

Theorem 2. *Assume that* $\sup_{i\geq 1+d} E\big(|v(\mathbf{X}_{i-1})\epsilon_i|^{2+\gamma_0}\big) < \infty$ *and* $E\big(|v(\widetilde{\mathbf{X}}_d)\epsilon_{1+d}|^{2+\gamma_0}\big) < \infty$ *hold and that Conditions 1, 2 and 3 also hold. Then under* $\mathcal{H}_0$, $R^*_{n,\psi} \to R^*_{\infty,\psi}$ *in distribution in the space* $\boldsymbol{D}_d[-\infty,\infty]$, *where* $R^*_{\infty,\psi}$ *is a centred Gaussian process with covariance function*

$$\begin{aligned} K^*_\psi(\mathbf{x},\mathbf{y}) &= K_\psi(\mathbf{x},\mathbf{y}) + \mathbf{G}'(\mathbf{x};\theta_0)\big(L_{1,1}(\theta_0) + 2\sum_{k=d}^{\infty} L_{1,k}(\theta_0)\big)\mathbf{G}(\mathbf{y};\theta_0) \\ &\quad -\mathbf{G}'(\mathbf{x};\theta_0)\sum_{k=d-1}^{\infty} E\big(v(\widetilde{\mathbf{X}}_d)\epsilon_{1+d}\mathbf{l}(\widetilde{\mathbf{X}}_k,\widetilde{X}_{k+1};\theta_0)\big) \\ &\quad -\mathbf{G}'(\mathbf{y};\theta_0)\sum_{k=d-1}^{\infty} E\big(v(\widetilde{\mathbf{X}}_d)\epsilon_{1+d}\mathbf{l}(\widetilde{\mathbf{X}}_k,\widetilde{X}_{k+1};\theta_0)\big), \end{aligned} \tag{15}$$

Corollary. *Under* $H_{1,n}$, *and the conditions of Theorem 1,* $R^*_{n,\psi} \to R^*_{\infty,\psi}$ *in distribution in the space* $\boldsymbol{D}_d[-\infty,\infty]$ *where* $R^*_{\infty,\psi}$ *is a Gaussian process with mean* $s(\mathbf{x})$ *and covariance function* $K^*_\psi(\mathbf{x},\mathbf{y})$ *defined in (15), where*

$$s(\mathbf{x}) = \int_{\mathbf{h}\leq\mathbf{x}} r(\mathbf{h})dF(\mathbf{h}) - \mathbf{G}(\mathbf{x};\theta_0)\int_{\mathbf{h}\leq\mathbf{x}}\int_{\mathbb{R}} \frac{r(\mathbf{h})}{v(\mathbf{h})}\mathbf{l}(\mathbf{h},y;\theta_0)d\widetilde{F}(\mathbf{h},y)$$

and $v(\cdot)$ *continuous are unknown.*

References

[1] Koul, H. and Stute, W. (1999), *Nonparametric model checks for time series*, Ann. Statist. 27, 204-236.
[2] Ngatchou-Wandji, J. (2002), *weak convergence of some marked empirical processes : Application to testing heteroscedasticity*, J. Nonparametr. Stat. 14, 325-339.
[3] Ngatchou-Wandji, J. (2008), *Local power of a Cramer-von Mises type test for parametric autoregressive models of order one*, Comput. Math. Appl. 56, 918-929.
[4] Ngatchou-Wandji, J. and Harel, M. (2013), *A Cramér-von Mises test for symmetry of the error distribution in asymptotically stationary stochastic models*, Stat. Inference Stoch. Process. 16, 207-236.