BlockShrink probability density estimator for dependent processes

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Résumé. On étudie le risque $\mathbb{L}_2$-integré, d'un estimateur BlockShrink de la densité de probabilité dans le cas de variables uniformément mélangeantes. En supposant que les coefficients de mélange uniforme $(\phi(i))_{i>0}$ sont arithmétiquement décroissants, on démontre que notre estimateur est adaptatif dans une classe d'espace de Sobolev de régularité inconnue.

Mots-clés. Estimation adaptative, densité, ondelettes, seuillage par block, processus ϕ-mélangeant, espace de Sobolev.

Abstract. We study the integrated $\mathbb{L}_2$-risk, of a wavelet BlockShrink density estimator based on dependent observations. We prove that the BlockShrink estimator is adaptive in class of Sobolev space with unknown regularity for uniformly mixing processes with arithmetically decreasing coefficients.

Keywords. Wavelet density estimation, Sobolev space, ϕ-mixing processes, Block thresholding.

1 Introduction

The functional estimation by the wavelet methods has been intensively used, these last years, in various areas. The popularity of these methods comes from the ease of their implementation, their flexibility, ability to catch details and for their high compression ratio. In the statistical literature, different types of wavelet estimators have been proposed. The performance of the first ones depended on the density's regularity. Later, adaptive procedures, as thresholding estimators, was developed to construct an estimate which does not depend on the explicit knowledge of this regularity. Those estimators are to make a fine selection of the coefficients estimators $\widehat{\beta}_{jk}$ of the wavelets coefficient β_{jk} and several thresholding techniques, including local, global and block thresholdings, have been developed.

The idea of block thresholding was introduced by Efroimovich (1985) as part of Fourier analysis. It has been adapted to the wavelet context analysis by Kerkyacharian et al.

(1996). The first localized block thresholding estimators has been developed by Hall et al. (1998, 1999), Cai (1996, 1997, 1999) and Chesneau (2008). The last one studied an $\mathbb{L}_p$ version of the BlockShrink estimator given by Cai (1996) in the iid case. Tribouley and Viennet (1998) explored the global thresholding method in β-mixing processes. To our knowledge, the BlockShrink estimator for the density model has not been studied for dependent processes.

The aim of this work is to extend some results, of BlockShrink estimator, to dependent processes in $\mathbb{L}_2$-norm for the density estimation. The function density f is supposed to belong to the Sobolev space $\mathbf{H}^s$ with compact support. We consider the ϕ-mixing's processes (Ibragimov (1962)) and we study the $\mathbb{L}_2$ error convergence for BlockShrink estimator f_n^b. We show that BlockShrink estimator is consistent with an optimal rate, under certain conditions on wavelet basis and mixing coefficients. Precisely we give the upper bound of the $\mathbb{L}_2$-mean error:

$$\mathbb{E}\|f - f_n^b\|_{\mathbb{L}_2}^2 := \mathbb{E}\left(\int_0^1 |f(t) - f_n(t)|^2 dt\right) \leq Cn^{-\frac{2s}{1+2s}}.$$

2 Wavelets and Sobolev space

We consider a wavelet basis on the interval $[0;1]$ of the form

$$\zeta = \{\varphi_{\tau k},\ \tau \geq 0; k = 0, \ldots, 2^j - 1; \psi_{jk},\ j \geq \tau; k = 0, \ldots, 2^j - 1\}$$

In general, $\varphi_{jk}(x)$ and $\psi_{jk}(x)$ are "periodic" or "boundary adjusted" dilation and translation of a "father" wavelet φ and a "mother" wavelet ψ, respectively. This last function is supposed to be N-regular.

For the sake of simplicity, we set $\varphi_{jk}(x) = 2^{j/2}\varphi(2^j x - k)$ and $\psi_{jk} = 2^{j/2}\psi(2^j x - k)$. Let τ be a sufficiently large integer, for any $j_0 \geq \tau$, a function f in $\mathbb{L}_2([0,1])$ can be expanded in a wavelet series as

$$f = \sum_{k=0}^{2^{j_0}-1} \alpha_{j_0 k}\varphi_{j_0 k} + \sum_{j \geq j_0} \sum_{k=0}^{2^j-1} \beta_{jk}\psi_{jk}, \tag{1}$$

where the wavelet coefficients are defined by

$$\alpha_{j_0 k} = \int f(x)\varphi_{j_0 k}(x)dx \quad \text{et} \quad \beta_{jk} = \int f(x)\psi_{jk}(x)dx.$$

Now, let us give a definition of Sobolev space, the main function spaces used in our study. Let $\beta_{\tau-1,k} = \alpha_{\tau k}$. We say that a function f belongs to the Sobolev space $\mathbf{H}^s$ if and only if the associated wavelet coefficients β_{jk}, satisfy

$$\left(\sum_{j\geq\tau-1} 2^{2js}\left(\sum_{k=0}^{2^j-1}|\beta_{jk}|^2\right)\right)^{1/2} < \infty,$$

where $s \in]0, N+1[$ with N denotes the wavelet regularity (see Meyer 1990).

3 BlockShrink estimator

Let $X_1, \ldots, X_n$, n observations from a ϕ-mixing's strictly stationary process, with unknown density f, relative to the Lebesgue measure on $\mathbb{R}$.
The sequence of the mixing coefficients $(\phi(l))_{l\geq 0}$ is assumed to have an arithmetic decay (ie : there exists $\theta > 1$ and a constant $c > 0$ such that for all $l \geq 1$ we have $\phi(l) \leq cl^{-\theta}$).
For a simplicity, we assume that $f \in \mathbb{L}_2([0.1])$, then f admits the following wavelet development representation (1) with $\alpha_{j_0k} = \mathbb{E}(\varphi_{j_0k}(X))$ and $\beta_{jk} = \mathbb{E}(\psi_{jk}(X))$, X of density f.
We define the non-linear BlockShrink estimator as proposed by Cai (1996) by:

$$f_n^b = \sum_{k=0}^{2^{j_0}-1} \widehat{\alpha}_{j_0k}\varphi_{j_0k} + \sum_{j=j_0}^{j_1}\sum_{K\in\mathcal{A}_j}\sum_{k\in\mathcal{B}_{jK}} \widehat{\beta}_{jk}\mathbb{I}_{\{\widehat{b}_{jK}\geq\lambda\}}\psi_{jk},$$

where j_0 is an integer chosen so that the linear variance term will not contribute to the overall error in the same spirit that the integer j_1 is chosen to make the bias term negligible in the overall error. We assume that $2^{j_1} \asymp n^{\frac{1}{2}}$ and $2^{j_0} \asymp (\log n)^{\varepsilon}$ with $\varepsilon > 2$. And for $j \in \{j_0, \ldots, j_1\}$, we define the sets

$$\mathcal{A}_j = \{1, \ldots, 2^j l_j^{-1}\} \text{ and } \mathcal{B}_{jK} = \{k \in \{0, \ldots, 2^j - 1\}; (K-1)l_j \leq k \leq Kl_j - 1\}, \text{ for } K \in \mathcal{A}_j,$$

where l_j is an increasing sequence in j such that $l_{j_0} \asymp (\log n)^{\varepsilon}$,

$$\widehat{b}_{jK} = \left(l_j^{-1}\sum_{k\in\mathcal{B}_{jK}}|\widehat{\beta}_{jk}|^2\right)^{\frac{1}{2}}$$

and the threshold λ will be precised bellow.

4 Main result

We have the following result:

Theorem 4.1 *We assume that f belongs to the class*

$$\mathcal{F}(M_1, M_2, B) = \{f \in \mathbf{H}^s, supp(f) \subset [B, -B], \|f\|_{\mathbb{H}^s} \leq M_1, \|f\|_\infty \leq M_2\},$$

then for $1/2 < s < N+1$, $\lambda = 4\sum_{l\geq 0}\phi(l)\|f\|_\infty/n^{1/2}$ *and for mixing coefficients arithmetically decreasing with* $\theta > 6 - \frac{4}{1+2(N+1)}$, *there exists a positive constant* C *such that*

$$\mathbb{E}\|f - f_n^b\|_2^2 \leq Cn^{-\frac{2s}{1+2s}}.$$

5 Preuves

Proofs are based on the work of Kerkyacharian et al. (1996), concentration inequality of Talagrand (1995) and on a bound of variance of partial sums of dependent r.v. in Viennet (1997).

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