A unified approach to the estimation of periodically integrated autoregressive models

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Résumé. Les tendances stochastiques et la périodicité sont des caractéristiques communes des séries temporelles, comme par exemple, en économie. Ces caractéristiques sont souvent entrelacées de sorte que les méthodes de la décomposition saisonnière traditionnelle, du lissage exponentiel et de la racine unité saisonnière (y compris les modèles ARIMA) ne sont pas entièrement satisfaisantes. Nous considérons une approximation basée sur des modèles périodiquement corrélés et périodiquement intégrés.

En utilisant le formulaire multi-compagnon du modèle autorégressif périodique et un paramétrage spectral, nous développons un cadre général pour les modèles qui sont périodiquement intégrés, ce qui permet d'avoir des modèles adéquats avec n'importe quelle configuration de racines unitaires, non-périodiques, saisonnières et périodiques. Comme nous travaillons directement avec les valeurs propres, nous pouvons directement imposer que certaines entre elles soient égales à 1 sans besoin ainsi d'imposer des restrictions non linéaires complexes sur les paramètres autorégressifs.

Mots-clés. Modèle autorégressif périodique, matrice multi-compagnon, paramétrage spectral, racines unitaires périodiques, intégration périodique.

Abstract. Stochastic trends and periodicity are common features of time series, for example in economics and business. These features are often intertwined in such a way that traditional seasonal decomposition, exponential smoothing and seasonal unit root (including ARIMA) methods are not always fully satisfactory. We consider an approach based on periodically correlated and periodically integrated models.

Using the multi-companion form of the periodic autoregressive model and a spectral parameterisation, we develop a general framework for periodically integrated models which allows for fitting models with any configuration of non-periodic, seasonal and periodic unit roots. Since we work directly with the eigenvalues, we are able to directly fix some of them to be equal to one, thus eliminating the need to impose complex non-linear restrictions on the autoregressive parameters.

Keywords. Periodic autoregressive model, multi-companion matrix, spectral parameterisation, periodic unit root, periodic integration

1 Extended Abstract

Stochastic trends and periodicity are common features of time series, for example in economics and business. These features are often intertwined in such a way that traditional seasonal decomposition, exponential smoothing and seasonal unit root (including ARIMA) methods are not always fully satisfactory. We consider an approach based on periodically correlated and periodically integrated models.

Periodic autoregressive (PAR) models have been widely used to model time series with periodic behaviour in wide variety of applications — economics (Franses and Paap, 2004), signal processing (Sakai, 1982) and hydrology (Hipel and McLeod, 1994), to name a few. Periodically integrated models with seasonal and/or periodic unit roots allow for flexible modelling of interacting non-periodic and periodic trends (Franses and Paap, 2004). Periodically correlated time series can be represented as multivariate stationary time series and vice versa (Gladyshev, 1961). This greatly facilitates the study and estimation of PAR models (Pagano, 1978). Similarly, periodically integrated processes can be converted to multivariate integrated processes. The configuration of the unit roots (Engle and Granger, 1987; Johansen, 1991) in the multivariate representation can then be related to stochastic trends in the original univariate time series (Boswijk et al., 1997).

Nevertheless, the scalar periodic filter representation is more parsimonious and easier to interpret. It also enhances estimation and testing, particularly when the periodic filters are factored into integrated and non-integrated components, in the spirit of the seasonal ARIMA models. For example, the periodic analogue of the difference filter $1 - B$ is a filter $1 - \alpha_s B$, with different coefficients for different seasons, such that $\prod_s \alpha_s = 1$ (B is the backward shift operator). This filter was first defined by Osborn (1988). Models incorporating this filter can be fitted by non-linear least squares. In the case of more unit roots or higher order integration, the non-linear algebraic relations between the parameters of unit root filters become cumbersome and depend not only on the number and configuration of these roots but also on the number of seasons. As a consequence, the existing literature is almost exclusively devoted to quarterly time series, i.e. number of seasons equal to four.

An alternative multivariate representation for periodic autoregressive models is the multi-companion representation introduced by Boshnakov (1997). This has the form of a vector autoregressive model of order one, whose autoregressive coefficient is a multi-companion matrix (Boshnakov, 2002). Matrices in this class have a special structure which allows for efficient parameterisation of their eigenspaces and Jordan decompositions and hence of the PAR models. We use the terms *spectral parameters* and *spectral parameterisation*. Periodic integration can be introduced by allowing some of the eigenvalues to have modulus one. For further details about the application of this parameterisation to periodic and multivariate models see Boshnakov and Iqelan (2009).

Using the multi-companion form of the PAR model and the spectral parameterisation, we develop a general framework for periodically integrated models which allows for fitting models with any configuration of non-periodic, seasonal and periodic unit roots. Since we work directly with the eigenvalues, we are able to directly fix some of them to be equal to one. This eliminates the need to impose complex non-linear restrictions on the standard autoregressive parameters, which need to be derived in the first place for each number of seasons and configuration of unit roots. For even more flexibility of modelling periodicity we can also fix the arguments of some eigenvalues to desired seasonal frequencies.

We discuss in more detail the case of quarterly periodicity (four seasons) which has been studied quite extensively (see Boswijk et al., 1997; Franses and Paap, 2004, and the references therein). We also give numerical examples.

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