Poisson QMLE for the multivariate integer-valued time series

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Résumé. Nous utilisons l'estimateur de quasi maximum de vraisemblance de Poisson (PQMLE) pour estimer, équation par équation, les paramètres des moyennes conditionnelles d'une série temporelle multivariée à valeurs entières. Des conditions de régularité sont données pour la consistance et la normalité asymptotique de cet estimateur. Des applications à des modèles particuliers, comme les modèles INAR et INGARCH multivariés, sont ainsi considérées. Des illustrations numériques, sur des simulations de Monte Carlo et sur des données réelles, sont fournies.

Mots-clés. Séries temporelles multivariées à valeurs entières, L'estimateur de quasi maximum de vraisemblance de Poisson, Consistance et normalité asymptotique, Copules, Moyenne conditionnelle.

Abstract. The Poisson quasi maximum likelihood estimator (PQMLE) is extended to consistently estimate, equation by equation, the conditional mean parameters of a multivariate time series of counts. regularity conditions for the consistency and the asymptotic normality of PQMLE are given. Applications to particular multivariate models, as multivariate INAR and INGARCH, are considered. Numerical illustrations via Monte Carlo simulations and real data applications, are provided.

Keywords. Multivariate time series of counts, Poisson quasi maximum likelihood, Consistency and asymptotic normality, Copulas, Conditional mean.

1 Introduction

The multivariate time series of counts are often found in a lot of applications in many scientific fields, for example, economy, biology and accidents analysis. Several models are proposed to deal with this kind of data. The most commonly used ones are the conditional means models whose the means are assumed to follow, conditionally on past observations, a vector autoregression. For example, the bivariate INGARCH model with bivariate Poisson conditional distribution Liu (2012) and the multivariate Double-Poisson INGARCH using copulas Heinen and Rengifo (2007). The study and the analysis of multivariate count time series pose several problems and questions. For instance, regarding to the conditional means models, the non negative integer-valued support multivariate distributions, which are able to accommodate the negative contemporaneous correlation between the series, are not abundantly available in the literature. One of these distributions which can be used to accommodate the negative contemporaneous correlation is the multivariate Poisson Log-Normal distribution Aitchison and Ho (1989), but it is difficult to be employed as a conditional distribution in the conditional mean models. However, This problem has been solved by using copulas which also still have limits to be used with the discrete distributions and require sometimes to apply the continuousation on the variables. In this paper, we propose a general and simple methodology to estimate such models. Where, we extend the Poisson quasi maximum likelihood estimator (PQMLE) studied by Ahmad and Francq (2015) to estimate the conditional means parameters of the multivariate time series of counts whatever the correlation between the series (positive, negative or absence of correlation) and whatever the nature of dispersion of the series (under-dispersed, over-dispersed or equi-dispersed). In this estimator, we need only to specify the conditional means of the series. The multivariate models will be estimated equation by equation (EbE) by giving general regularity conditions under which the EbE-PQMLE is consistent and asymptotically normal. Section 2 contains the general formulation of the model. Section 3 shows the main results concerning the asymptotic behavior of the EbE-PQMLE. Section 4 contains an application of EbE-PQMLE to the Bivriate Poisson INAR(1) model and an example of its concerned Monte Carlo simulation results. Section 5 concludes, and the theatrical assumptions are collected in Section 5.

2 General formulation

Assume that X_t is k-dimensional vector of a time series of counts valued in $\mathbb{N}^k$, assume also that the conditional mean of each component of $X_t = (X_{1.t}, \dots, X_{k.t})'$ are equal to another vector $\Lambda_t = (\lambda_{1.t}, \dots, \lambda_{k.t})'$ such that

$$E\left(X_{d.t} \mid X_{d.u}, u < t\right) = \lambda_d\left(X_{d.t-1}, X_{d.t-2}, \dots, X_{c.t-1}, X_{c.t-2}, \dots, ; \theta_{d.0}\right) = \lambda_{d.t}. \tag{2.1}$$

Where d and $c \in \{1, ..., K\}$, and $d \neq c$. $\theta_{d.0}$ is an unknown parameter belonging to some parameter space Θ_d, and

$$\lambda_d \text{ is a measurable function valued in } (\underline{\omega}, +\infty) \text{ for some } \underline{\omega} > 0 \tag{2.2}$$

This formulation allows to λ_d to be different from the equation to the other. For example, λ_d can be a linear function while λ_c is a log-linear or non-linear function. We assume also that the marginal distribution has a moment slightly greater than 1

$$EX_{d.t}^{1+\varepsilon} < \infty, \text{ for some } \varepsilon > 0, \tag{2.3}$$

which entails the existence of the conditional mean.

3 Estimating the conditional means parameters

We consider the special case of the model (2.1) when the conditional mean $\lambda_{d.t}$ depends on the past values of the components of the vector X_t and only on its past values. To estimate this model, we will use the PQMLE studied by Ahmad and Francq (2015) and Christou and Fokianos (2013) for estimating, separately, the conditional mean parameters of each series, which is called equation-by-equation estimation (EbEE). The reader is referred to Francq and Zakoian (2014) for more details on estimation of multivariate GARCH models using EbEE method. At first, consider that $\theta_{d.0}$, for $d \in \{1, ...K\}$, is an unknown parameter belonging to some parameter space Θ_d. The EbE-PQMLE is defined as a solution of the following problem of maximization

$$\widehat{\theta}_{d.n} = \arg \max_{\theta_{d.n} \in \Theta_d} \widetilde{L}_{d.n}(\theta_d), \quad \widetilde{L}_{d.n}(\theta_d) = \frac{1}{n} \sum_{t=s+1}^{n} \widetilde{\ell}_{d.t}(\theta_d), \tag{3.1}$$

where $\widetilde{\ell}_{d.t}(\theta) = -\widetilde{\lambda}_{d.t}(\theta) + X_{d.t} \log \widetilde{\lambda}_{d.t}(\theta)$. $\widetilde{\lambda}_{d.t}(\theta)$ and $X_{d.t}$ are respectively the dth components of $\widetilde{\Lambda}_t(\theta)$ and X_t. $\widetilde{\Lambda}_t(\theta)$ is obtained by setting some initial values $X_0, X_{-1}..$ involved in $\lambda_{d.t}(\theta)$. s is a constant for reducing the effects of the initial values and its value is asymptotically unimportant. The regularity conditions required for EbE-PQMLE are slightly different from those of PQMLE. The technical assumptions of consistence (A1-A7) and asymptotic normality (A8-A12) are given in the Appendix.

Theorem 3.1. Let $X_{d.t}$ be a stationary and ergodic process defined in (2.1) satisfying (2.2) and (2.3) as well as the assumptions A1-A7. Let $\widehat{\theta}_d$ be the EbE-PQMLE, then

$$\widehat{\theta}_{d.n} \to \theta_{d.0} \quad a.s. \qquad \text{as } n \to \infty.$$

For the asymptotic normality, we assume the existence of the conditional variance of $X_{d.t}$ given its past, such that

$$E\left(X_{d.t}^2 \mid X_{d.u}, u < t\right) := v_{d.t}(\theta_{d.0}) + \lambda_{d.t}^2(\theta_{d.0}). \tag{3.2}$$

Theorem 3.2. Assume that $(X_{d,t})$ satisfies the conditions of Theorem 3.1. Assume also (3.2) and A8-A11 are hold. Then

$$\sqrt{n}(\widehat{\theta}_{d.n} - \theta_{d.0}) \xrightarrow{d} \mathcal{N}\left(0, \Sigma := J_{dd}^{-1} I_{dd} J_{dd}^{-1}\right) \qquad \text{as } n \to \infty.$$

Where

$$J = E \frac{1}{\lambda_{d.t}(\theta_{d.0})} \frac{\partial \lambda_{d.t}(\theta_{d.0})}{\partial \theta_d} \frac{\partial \lambda_{d.t}(\theta_{d.0})}{\partial \theta_d'}, \quad I = E \frac{v_{d.t}(\theta_{d.0})}{\lambda_{d.t}^2(\theta_{d.0})} \frac{\partial \lambda_{d.t}(\theta_{d.0})}{\partial \theta_d} \frac{\partial \lambda_{d.t}(\theta_{d.0})}{\partial \theta_d'}. \tag{3.3}$$

It can be shown that, under the assumptions of Theorem 3.2, the asymptotic variance of the EbE-PQMLE can be consistently estimated by $\widehat{\Sigma}_{dd} = \widehat{J}_{dd}^{-1}\widehat{I}_{dd}\widehat{J}_{dd}^{-1}$ with

$$\widehat{J}_{dd} = \frac{1}{n}\sum_{t=s+1}^{n} \frac{1}{\widetilde{\lambda}_{d.t}(\widehat{\theta}_{d.n})} \frac{\partial\widetilde{\lambda}_{d.t}(\widehat{\theta}_{d.n})}{\partial\theta} \frac{\partial\widetilde{\lambda}_{d.t}(\widehat{\theta}_{d.n})}{\partial\theta'}, \tag{3.4}$$

$$\widehat{I}_{dd} = \frac{1}{n}\sum_{t=s+1}^{n} \left(\frac{X_{d.t}}{\widetilde{\lambda}_{d.t}(\widehat{\theta}_{d.n})} - 1\right)^2 \frac{\partial\widetilde{\lambda}_{d.t}(\widehat{\theta}_{d.n})}{\partial\theta} \frac{\partial\widetilde{\lambda}_{d.t}(\widehat{\theta}_{d.n})}{\partial\theta'}. \tag{3.5}$$

4 Application of EbE-PQMLE to the bivariate Poisson INAR(1) model

One of the most popular multivariate count time series model is the multivariate INAR(1) (M-INAR(1)) model, which is introduced by Franke and Subba Rao (1995). The M-INAR(1) model defines X_t as a K-dimensional vector of a non negative integer-valued in $\mathbb{N}^k$, where

$$X_t = \Phi \circ X_{t-1} + E_t$$

Φ is a $k \times k$ matrix with entries $\alpha_{ij} \in \{0, 1\}$, for $i, j = 1, ..., K$. $\Phi\circ$ acts as the usual matrix multiplication keeping in the same time the properties of the binomial thinning operation. The innovation term, $E_t = (\epsilon_{1t}, ..., \epsilon_{Kt})'$, is assumed to be iid $\mathbb{N}^k$-valued random vector whose the components have finite mean and variance ω_d and σ_d^2 respectively. Let assume that K=2 and the joint probability mass function of the two innovation processes $(\epsilon_{1t}, \epsilon_{2t})$ is a bivariate Poisson distribution. We denote this distribution as $\mathcal{BP}(\lambda_{\epsilon_1}, \lambda_{\epsilon_2}, \nu)$. The reader is referred to Pedeli and Karlis (2013) for more details on the properties and the estimation of BP-INAR(1) model. One can note that by taking the conditional expectation of the BP-INAR(1) process, we will have a model similar to that defined in (2.1), where $E(X_t|X_u, u < t) = \Lambda_t = \Omega + \Phi X_{t-1}$ and $E(E_1) = (\lambda_{\epsilon_1} + \nu, \lambda_{\epsilon_2} + \nu)' = \Omega$.

4.1 Example of Monte Carlo simulation results

Table 1 shows the results of Monte Carlo simulations for the bivariate Poisson INAR(1) model. The number of simulations is N=1000. For each simulation, the bivariate model is estimated, equation by equation, using (3.1). The means of the estimated values of θ_0 are given in the rows "$\widehat{\theta}$". This table also gives two different estimators of the root-mean-square deviation $\sqrt{E\left(\widehat{\theta}_n - \theta_0\right)^2}$: the empirical standard errors (ESE) and the estimated standard error based on the asymptotic theory (ASE). Table 1 show that the means of the estimated parameters are satisfactorily close to their theoretical values, especially for large sample sizes. Moreover the two estimations of the standard deviations, the ESE and ASE , are very similar.

Table 1: The finite sample behaviour of EbE-PQMLE for the Bivariate INAR model

		BP-INAR(1), $E_t \sim \mathcal{BP}(2,3,2)$					
n		$\omega_{1.0}$=4	$\alpha_{11.0}$=0.2	$\alpha_{12.0}$=0.4	$\omega_{2.0}$=5	$\alpha_{22.0}$=0.3	$\alpha_{21.0}$=0.4
500	$\widehat{\theta}$	4.047	0.199	0.398	5.070	0.295	0.399
	ESE	0.544	0.042	0.038	0.579	0.042	0.043
	ASE	0.544	0.041	0.038	0.591	0.041	0.045
1000	$\widehat{\theta}$	4.045	0.197	0.399	5.039	0.298	0.400
	ESE	0.395	0.030	0.027	0.408	0.029	0.032
	ASE	0.386	0.029	0.027	0.417	0.029	0.032

5 Conclusion

EbE-PQMLE provides a general way for estimating the conditional mean parameters of the multivariate time series of counts. If the conditional means are correctly specified, under some regularity conditions, the EbE-PQMLE is consistent and asymptotically normal. The results of this work can be applied to a large variety of counts time series models, as the multivariate INGARCH and multivariate INAR models.

6 Appendix

For the consistency of EBE-PQMLE we assume that

A1 We have $\theta_{d.0} \in \Theta_d$ where Θ_d is compact.

A2 The process $X_{d.t}$ is stationary and ergodic.

A3 $\lambda_{d.1}(\theta_d) = \lambda_{d.1}(\theta_{d.0})$ almost surely if and only if $\theta_d = \theta_{d.0}$

A4 $\widetilde{\lambda}_{d.t}(\theta_d) > \underline{\omega}$, for some $\underline{\omega} > 0$.

The next two assumptions are for demonstrate that the initial values have no effects on the asymptotic properties of EBE-PQMLE.

A5 $\lim_{t\to\infty} a_{d.t} = 0$ and $\lim_{t\to\infty} X_{d.t}a_{d.t} = 0$, where $a_{d.t} = \sup_{\theta_d\in\Theta_d} \left|\widetilde{\lambda}_{d.t}(\theta_d) - \lambda_{d.t}(\theta_d)\right|$,

A6 $\lim_{n\to\infty} \sup_{\theta_d\in V(\Theta_d)} \left|\widetilde{L}_{d.n}(\theta_d) - L_{d.n}(\theta_d)\right| = 0, a.s.$

A7 $E\left|\ell_{d,1}(\theta_d)\right| < \infty$ and if $\theta_d \neq \theta_{d.0}, E\ell_{d,1}(\theta_d) < E\ell_{d,1}(\theta_{d.0})$

A8 any $\theta_d \neq \theta_{d.0}$ has neighbourhood $V(\theta_d)$ such that $\limsup_{n\to\infty} \sup_{\theta^*\in V(\Theta_d)} \widetilde{L}_{d.n}(\theta_d) < \liminf_{n\to\infty} \widetilde{L}_{d.n}(\theta_{d.0})$

For the asymptotic normality

A9 If $\theta_{d0} \in \overset{\circ}{\Theta}_d$, where $\overset{\circ}{\Theta}$ denotes the interior of Θ.

A10 The existence of continuous second-order derivatives for $\lambda_{d.t}$ and $\widetilde{\lambda}_{d.t}$

The next assumption is introduced to handle initial values.

A11 $b_{d.t}, b_{d.t}X_{d.t}$ and $a_{d.t}d_{d.t}X_{d.t}$ are of order $O(t^{-\kappa})$ for some $\kappa > 1/2$, where

$$b_{d.t} = \sup_{\theta_d \in \Theta_d} \left\| \frac{\partial \widetilde{\lambda}_{d.t}(\theta_d)}{\partial \theta_d} - \frac{\partial \lambda_{d.t}(\theta_d)}{\partial \theta_d} \right\|.$$

$$d_{d.t} = \sup_{\theta_d \in \Theta_d} \max \left\{ \left\| \frac{1}{\lambda_{d.t}(\theta_d)} \frac{\partial \lambda_{d.t}(\theta_d)}{\partial \theta_d} \right\|, \left\| \frac{1}{\widetilde{\lambda}_{d.t}(\theta_d)} \frac{\partial \widetilde{\lambda}_{d.t}(\theta_d)}{\partial \theta_d} \right\| \right\}$$

A12 The matrix J and I are existent and J is invertible

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