Estimating new multivariate risk measures

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Résumé. Adrian et Brunnermeier (2011) ont proposé une mesure du risque dans le but de quantifier le risque systémique dans le système financier. Cette mesure est appelée CoVaR (Value-at-Risk conditionnelle). La CoVaR mesure la contribution d'une institution financière au risque systémique et sa contribution au risque d'autres institutions financières. Le problème de mesures du risque a été traditionnellement traité dans une version univarié (voir par exemple la Value-at-Risk). En revanche de nombreuses extensions multidimensionnelles ont été étudiées dans la dernière décennie. Deux extensions multivariées de la mesure classique CoVaR univariée sont définies dans ce travail. Ces nouvelles mesures multivariées sont basées sur les ensembles de niveau de fonctions de distribution multidimensionnelles (resp. survie multivariées). Plusieurs propriétés importantes des nouvelles mesures de risque multivariées sont fournies. En particulier : "elicitability", propriété d'invariance et de dépendance sont examinées. Les expressions pour les CoVaRs multivariées proposées, peuvent être facilement calculées dans la classe des copules d'Archimède. Une procédure d'estimation semi-paramétrique est présentée pour les mesures proposées. Les estimateurs sont obtenus à partir des expressions des CoVaRs multivariées dans des conditions de copule d'Archimède. En outre, ils sont construits en utilisant un estimateur semi-paramétrique du générateur associé à la copule (resp. copule de survie) et l'estimation empirique de la fonction quantile. La performance des estimateurs définis est étudiée en considérant différents modèles de données simulées. Enfin, les estimateurs des mesures de CoVaR multivariée sont appliqués dans un cas réel d'assurance.

Mots-clés. Copules et dépendance, ensembles de niveau des fonctions de distribution, mesures de risque multivaries, ordres stochastiques, Value-at-Risk.

Abstract. Adrian and Brunnermeier (2011) proposed a risk measure with the purpose of quantifying the systemic risk in the financial system. This measure is called CoVaR

(Conditional Value-at-Risk). CoVaR measures a financial institution's contribution to systemic risk and its contribution to the risk of other financial institutions. In spite of the fact that the problem of measuring risk market has been traditionally handled in a univariate version, by Value-at-Risk, many multidimensional extensions have been investigated in the last decade. Two multivariate extensions of the classic univariate CoVaR are defined in this work. These new multivariate measures are based on the level sets of multivariate distribution functions (*resp.* of multivariate survival distribution). Several important properties of the new multivariate risk measures are provided. Particularly, elicitability, invariance and comonotonic dependence properties are examined. Interestingly, easily computed expressions for the multivariate CoVaRs are given in the class of Archimedean copulas. The aim of this work is focussed on the estimation of the multivariate CoVaR measures. A semiparametric estimation procedure is presented for the proposed multivariate risk measure. The estimators are obtained from the expressions of the multivariate CoVaRs under Archimedean copula conditions. Furthermore, they are constructed by using a semiparametric estimator of the generator associated with the copula (*resp.* survival copula) and the empirical estimation of the quantile function. The performance of the defined estimators is studied by considering different models of simulated data. Finally, the estimators of the multivariate CoVaR measures are calculated in an insurance real case.

Keywords. Copulas and dependence, Level sets of distribution functions, Multivariate risk measures, Stochastic orders, Value-at-Risk.

1 Preliminaries and Definitions

Measuring the market risk gets involved in the risk-based methodology for supervise and regular the financial sector which is gaining ground in emerging and industrial countries. Traditionally, this problem has been handled in a univariate version. However, much research have studied risk measures in a multivariate setting in the last decade (see Embrechts and Puccetti (2006), Nappo and Spizzichino (2009), Prékopa (2012), Cousin and Di Bernardino (2013) and Cousin and Di Bernardino (2014)).

On the other hand, the necessity of measuring external risks, such as the systemic risk, emerges as a consequence of the solvability of the financial institutions could be affected by these risks. CoVaR (Conditional Value-at-Risk) is a systemic risk measure defined by Adrian and Brunnermeier (2011).

In Definitions 1.1 and 1.2, two new generalizations of the classic univariate CoVaR are proposed by using the level sets approach in Cousin and Di Bernardino (2013) and Cousin and Di Bernardino (2014).

Assume that $\mathbf{X} = (X_1, \ldots, X_d)$ is a non-negative absolutely-continuous random vector (with respect to Lebesgue measure λ on $\mathbb{R}^d$) with distribution function F and survival function $\overline{F}$. In addition, the multivariate distribution function F is assumed to be partially strictly-increasing such that $E(X_i) < \infty$ for $i = 1, \dots, d$. From now on, the above conditions will be called *regularity conditions*.

Definition 1.1 (Multivariate Lower-Orthant CoVaR) *Consider a random vector* $\mathbf{X}$ *which satisfies the regularity conditions. For* $\alpha \in (0,1)$*, we define the multivariate lower-orthant CoVaR at probability level* α *by*

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_{\boldsymbol{\omega}}(\mathbf{X}|\mathbf{X} \in \partial \underline{L}(\alpha)) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(X_1|\mathbf{X} \in \partial \underline{L}(\alpha)) \\ \vdots \\ \mathrm{VaR}_{\omega_d}(X_d|\mathbf{X} \in \partial \underline{L}(\alpha)) \end{pmatrix},$$

where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d)$ *is a marginal risk vector with* $\omega_i \in [0,1]$*, for* $i = 1, \dots, d$*, and* $\partial \underline{L}(\alpha)$ *is the boundary of the set* $\underline{L}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^d_+ : F(\boldsymbol{x}) \geq \alpha\}$*. Hence, under regularity conditions,*

$$\underline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(X_1|F(\mathbf{X}) = \alpha) \\ \vdots \\ \mathrm{VaR}_{\omega_d}(X_d|F(\mathbf{X}) = \alpha) \end{pmatrix}.$$

Correspondingly, the multivariate upper-orthant CoVaR is introduced.

Definition 1.2 (Multivariate Upper-Orthant CoVaR) *Consider a random vector* $\mathbf{X}$ *which satisfies the regularity conditions. For* $\alpha \in (0,1)$*, we define the multivariate upper-orthant CoVaR at probability level* α *by*

$$\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_{\boldsymbol{\omega}}(\mathbf{X}|\mathbf{X} \in \partial \overline{L}(\alpha)) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(X_1|\mathbf{X} \in \partial \overline{L}(\alpha)) \\ \vdots \\ \mathrm{VaR}_{\omega_d}(X_d|\mathbf{X} \in \partial \overline{L}(\alpha)) \end{pmatrix},$$

where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d)$ *is a marginal risk vector with* $\omega_i \in [0,1]$*, for* $i = 1, \dots, d$*, and* $\partial \overline{L}(\alpha)$ *is the boundary of the set* $\overline{L}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^d_+ : \overline{F}(\boldsymbol{x}) \leq 1 - \alpha\}$*. Hence, under regularity conditions,*

$$\overline{\mathrm{CoVaR}}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \begin{pmatrix} \mathrm{VaR}_{\omega_1}(X_1|\overline{F}(\mathbf{X}) = 1 - \alpha) \\ \vdots \\ \mathrm{VaR}_{\omega_d}(X_d|\overline{F}(\mathbf{X}) = 1 - \alpha) \end{pmatrix}.$$

Important properties of the two new multivariate risk measures have been studied. For more details see Di Bernardino et al. (2015). Furthermore, characterizations and estimations of these new risk measures are provided in the class of Archimedean copula (Sections 2 and 3).

2 Multivariate CoVaRs under Archimedean copulas

In this section, analytical expressions for multivariate CoVaR measures when **X** follows an Archimedean copula (or survival copula) are presented. These expressions come from the definition of Archimedean copulas (see Nelsen (2006)) and the stochastic representation of an Archimedean copula given in McNeil and Něslehová (2009).

Corollary 2.1 *Let* **X** *be a d-dimensional random vector with an Archimedean copula with generator ϕ and $\alpha \in (0,1)$. Therefore,*

$$\underline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_{\omega_i}\left[F^{-1}_{X_i}(\phi^{-1}(S_i\phi(\alpha)))\right], \ \textit{for } i = 1, \ldots, d, \tag{1}$$

where $\boldsymbol{\omega} \in [0,1]^d$ and S_i is a random variable with $Beta(1, d-1)$ distribution.

Corollary 2.2 *Let* **X** *be a d-dimensional random vector with an Archimedean survival copula with generator ϕ and $\alpha \in (0,1)$. Therefore,*

$$\overline{\mathrm{CoVaR}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \mathrm{VaR}_{\omega_i}\left[\overline{F}^{-1}_{X_i}(\phi^{-1}(S_i\phi(1-\alpha)))\right] \ \textit{for } i = 1, \ldots, d, \tag{2}$$

where $\boldsymbol{\omega} \in [0,1]^d$ and S_i is a random variable with $Beta(1, d-1)$ distribution.

3 An estimation procedure for multivariate CoVaRs

Assuming that **X** has an Archimedean copula structure, a semiparametric estimator for the multivariate lower CoVaR is given from Equation (1). A maximum pseudo-likelihood estimator of the dependence parameter θ associated with the generator of the copula and the empirical quantile estimation are considered to construct this semiparametric estimator.

Definition 3.1 *Let* ***X*** *be a $d-$dimensional random vector with Archimedean copula with generator ϕ_θ and $\alpha \in (0,1)$. A semiparametric estimator of the $i-$component of the multivariate lower* CoVaR *is defined as*

$$\widehat{\underline{\mathrm{CoVaR}}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \widehat{\mathrm{VaR}}_{\omega_i}\left[\hat{F}^{-1}_{X_i}(\phi^{-1}_{\hat{\theta}_n}(S_i\phi_{\hat{\theta}_n}(\alpha)))\right], \quad \textit{for } i = 1, \ldots, d,$$

where $\boldsymbol{\omega} \in [0,1]^d$, S_i is a random variable with $Beta(1, d-1)$ distribution, $\widehat{\mathrm{VaR}}_\omega(X)$ is the empirical estimator of $\mathrm{VaR}_\omega(X)$, $\phi_{\hat{\theta}_n}$ is the semiparametric estimator of ϕ_θ and $\hat{F}^{-1}_{X_i}$ is the empirical estimator of $F^{-1}_{X_i}$ for $i = 1, \ldots, d$.

Now, let assume that **X** has an Archimedean survival copula structure. From Equation (2), a semiparametric estimator for multivariate upper CoVaR is presented in Definition 3.2. This estimator depends on a semiparametric estimator of the generator of the Archimedean survival copula and the empirical estimation of the quantile functions.

Definition 3.2 *Let* $\mathbf{X}$ *be a* $d-$*dimensional random vector with Archimedean survival copula with generator* ϕ_θ *and* $\alpha \in (0,1)$*. A semiparametric estimator of the* $i-$*component of the multivariate upper* CoVaR *is defined as*

$$\widehat{\overline{\mathrm{CoVaR}}}^{i}_{\alpha,\boldsymbol{\omega}}(\mathbf{X}) = \widehat{\mathrm{VaR}}_{\omega_i}\left[\hat{\overline{F}}^{-1}_{X_i}(\phi^{-1}_{\hat{\theta}_n}(S_i\phi_{\hat{\theta}_n}(1-\alpha)))\right], \quad \text{for } i = 1,\ldots,d,$$

where $\boldsymbol{\omega} \in [0,1]^d$*,* S_i *is a random variable with* $Beta(1, d-1)$ *distribution,* $\widehat{\mathrm{VaR}}_\omega(X)$ *is the empirical estimator of* $\mathrm{VaR}_\omega(X)$*,* $\phi_{\hat{\theta}_n}$ *is the semiparametric estimator of* ϕ_θ *and* $\hat{\overline{F}}^{-1}_{X_i}$ *the empirical estimator of* $\overline{F}^{-1}_{X_i}$ *for* $i = 1,\ldots,d$*.*

Two different copula models are simulated to evaluate the performance of the previous introduced estimator. We consider the estimator of the lower multivariate CoVaR measure given in Definition 3.1 (the estimator proposed in Definition 3.2 could be similarly studied) and we restrict ourselves to the bivariate case (the illustrations could be extended to any dimension). Boxplots of the ratio $\widehat{\underline{\mathrm{CoVaR}}}^{1}_{\alpha,\boldsymbol{\omega}}(X,Y)/\underline{\mathrm{CoVaR}}^{1}_{\alpha,\boldsymbol{\omega}}(X,Y)$ are illustrated for the two simulated models, different values of α and ω, and two different sample sizes. Although the simulated data set is generated from Gumbel and Ali-Mikhail-Haq copulas, the semiparametric estimator of the generator of the copula to calculate the above ratio is also obtained from others copula models. The aim is to analyse misspecification model error to study the bias and the variance of the estimation when the parametric form of the copula is not appropriate to the data. Moreover, the empirical standard deviation and the relative mean square error of the estimator of the multivariate lower CoVaR are analysed when the values of α, ω and sample size vary.
The introduced estimators for multivariate CoVaRs measures in Definitions 3.1 and 3.2 are calculated for an insurance real data set, **Loss-ALAE data** (in the log scale). The data set contains $n = 1500$ claims. Each claim consists of an indemnity payment (the loss, X) and an allocated loss adjustment expense (ALAE, Y). Loss-ALAE data in the log scale, the respective semiparametric estimated α-level sets ($\partial\underline{L}(\alpha)$ and $\partial\overline{L}(\alpha)$), the univariate empirical quantiles of Loss-ALAE data, and the estimators of multivariate CoVaRs measures are gathered in Figure 1.
Some results of quantile regression estimations by using extreme theory for the two multivariate CoVaRs will be given. These results are currently in preparation.

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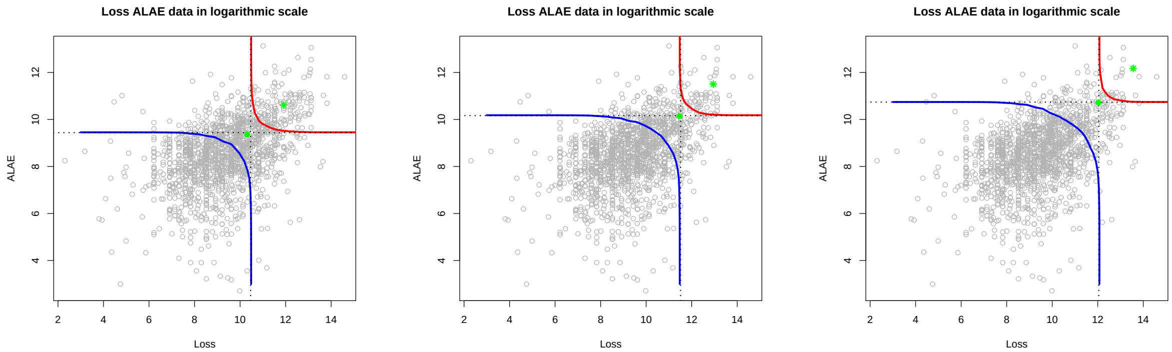

Figure 1: Loss ALAE data in log scale, boundary of estimated level sets ($\partial \underline{L}(\alpha)$, red line), boundary of estimated level sets ($\partial \overline{L}(\alpha)$, blue line), empirical quantile of Loss data (dotted black line), empirical quantile of ALAE data (dotted black line), $\underline{\widehat{\text{CoVaR}}}_{\alpha,\omega}$ (stars) and $\overline{\widehat{\text{CoVaR}}}_{\alpha,\omega}$ (solid circles) with ($\alpha = 0.75$, $\omega = 0.9$) (left panel); ($\alpha = 0.9$, $\omega = 0.95$) (center panel); ($\alpha = 0.95$, $\omega = 0.98$) (right panel).

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