QML INFERENCE FOR VOLATILITY MODELS WITH COVARIATES

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Résumé. La loi asymptotique de l'estimateur du quasi-maximum de vraisemblance gaussien est établie pour la vaste classe des modèles GARCH asymétriques avec covariables exogènes. La vraie valeur du paramètre n'est pas contrainte à se situer à l'intérieur de l'espace des paramètres, ce qui nous permet de développer des tests de significativité des paramètres. En particulier, la pertinence des variables exogènes peut être évaluée. Les résultats sont obtenus sans faire l'hypothèse que les innovations sont indépendantes, ce qui permet de prendre en compte différents ensembles d'information. Des expériences de Monte Carlo et des applications sur séries financières illustrent les résultats asymptotiques. En particulier, une étude empirique montre que la volatilité réalisée est une covariable utile pour prévoir les carrés des rendements, mais ne constitue pas un proxy idéal de la volatilité.

Mots-clés. Bord de l'espace des paramètres, Convergence forte et loi asymptotique de l'estimateur du quasi-maximum de vraisemblance gaussien, Loi asymptotique non normale, Modèle APARCH avec variables explicatives, Modèle GARCH-X.

Abstract. The asymptotic distribution of the Gaussian quasi-maximum likelihood estimator (QMLE) is obtained for a wide class of asymmetric GARCH models with exogenous covariates. The true value of the parameter is not restricted to belong to the interior of the parameter space, which allows us to derive tests for the significance of the parameters. In particular, the relevance of the exogenous variables can be assessed. The results are obtained without assuming that the innovations are independent, which allows conditioning on different information sets. Monte Carlo experiments and applications to financial series illustrate the asymptotic results. In particular, an empirical study demonstrates that the realized volatility is an helpful covariate for predicting squared returns, but does not constitute an ideal proxy of the volatility.

Keywords. APARCH model augmented with explanatory variables, Boundary of the parameter space, Consistency and asymptotic distribution of the Gaussian quasi-maximum likelihood estimator, GARCH-X models, Non-normal asymptotic distribution, Power-transformed and Threshold GARCH with exogenous covariates.

1 Introduction

The GARCH-type models are of the form

$$\varepsilon_t = \sigma_t \eta_t, \tag{1.1}$$

where the squared volatility σ_t^2 is the best predictor of ε_t^2 given a certain information set $\mathcal{F}_{t-1}$ available at time t. More precisely, it is assumed that $E(\varepsilon_t^2 \mid \mathcal{F}_{t-1}) = \sigma_t^2 > 0$, or equivalently that $\sigma_t > 0$, $\sigma_t \in \mathcal{F}_{t-1}$ and $E(\eta_t^2 \mid \mathcal{F}_{t-1}) = 1$. For the usual GARCH models, $\mathcal{F}_{t-1}$ is simply the sigma-field generated by the past returns $\{\varepsilon_u, u < t\}$, and the volatility has a parametric form $\sigma_t = \sigma(\varepsilon_u, u < t; \boldsymbol{\theta}_0)$. It is however often the case that some extra information is available, under the form of a vector $\boldsymbol{x}_{t-1}$ of exogenous covariates, such as the daily volume of transactions, or high frequency intraday data, or even series of other returns. It is natural to try to take advantage of the extra information, in order to improve the prediction of the squares. To incorporate the information conveyed by $\{\boldsymbol{x}_u, u < t\}$ into $\mathcal{F}_{t-1}$, researchers have considered GARCH models augmented with additional explonatory variables, the so-called GARCH-X models, which are of the form $\sigma_t = \sigma(\varepsilon_u, \boldsymbol{x}_u, u < t; \boldsymbol{\vartheta}_0)$.
Let $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. We consider the model defined by

$$\begin{cases} \varepsilon_t = h_t^{1/\delta} \eta_t \\ h_t = \omega_0 + \sum_{i=1}^q \alpha_{0i+}(\varepsilon_{t-i}^+)^\delta + \alpha_{0i-}(\varepsilon_{t-i}^-)^\delta + \sum_{j=1}^p \beta_{0j} h_{t-j} + \boldsymbol{\pi}_0' \boldsymbol{x}_{t-1} \end{cases} \tag{1.2}$$

where $\boldsymbol{x}_t = (x_{1,t}, \ldots, x_{r,t})'$ is a vector of r exogenous covariates. To ensure that $h_t > 0$ with probability one, assume that the covariates are almost surely positive and that the coefficients satisfy $\alpha_{0i+} \geq 0$, $\alpha_{0i-} \geq 0$, $\beta_{0j} \geq 0$, $\omega_0 > 0$, $\delta > 0$ and $\boldsymbol{\pi}_0 = (\pi_{01}, \ldots, \pi_{0r}) \geq 0$ componentwise.

Our first objective is to study the asymptotic distribution of the QMLE of the APARCH-X model when the parameter is not restricted to belong to the interior of the parameter space.

Our second objective is to propose tests of nullity for one or several components of $\boldsymbol{\vartheta}_0$. This is closely related to the first objective because, due to the positivity constraints on the components of $\boldsymbol{\vartheta}_0$, under the null the true parameter stands at the boundary of the parameter space.

2 Main results

2.1 Strict stationarity

In this subsection, we are interested in the strictly stationary solutions which will be the main condition for the consistency of the QMLE. Assuming that $p \geq 2$ and $q \geq 2$, let the vector of dimension $2q+p-2$

$$\boldsymbol{Y}_t = \left(h_{t+1}, \ldots, h_{t-p+2}, \left(\varepsilon_t^+\right)^\delta, \left(\varepsilon_t^-\right)^\delta, \ldots, \left(\varepsilon_{t-q+2}^+\right)^\delta, \left(\varepsilon_{t-q+2}^-\right)^\delta\right)'.$$

It is easy to see that (ε_t) satisfies (1.2) if and only if

$$\boldsymbol{Y}_t = \boldsymbol{C}_{0t}\boldsymbol{Y}_{t-1} + \boldsymbol{B}_{0t}, \tag{2.1}$$

where $\boldsymbol{B}_{0t} = (\omega_0 + \boldsymbol{\pi}_0'\boldsymbol{x}_t, 0, \ldots, 0)'$ is a vector of dimension $2q+p-2$ and $\boldsymbol{C}_{0t}$ is a matrix depending on $(\eta_t^+)^\delta$, $(\eta_t^-)^\delta$ and

$$\boldsymbol{\vartheta}_0 = (\boldsymbol{\theta}_0', \boldsymbol{\pi}_0')', \qquad \boldsymbol{\theta}_0 = (\omega_0, \alpha_{01+}, \alpha_{01-}, \ldots, \alpha_{0q+}, \alpha_{0q-}, \beta_{01}, \ldots, \beta_{0p})'.$$

Now assume that

A1: $(\eta_t, \boldsymbol{x}_t')$ is a strictly stationary and ergodic process, and there exists $s>0$ such that $E|\eta_1|^s < \infty$ and $E\|\boldsymbol{x}_1\|^s < \infty$.

The stationarity relies on the top Lyapunov $\gamma := \lim_{t\to\infty} \frac{1}{t}\log\|\boldsymbol{C}_{0t}\boldsymbol{C}_{0,t-1}\cdots\boldsymbol{C}_{01}\|$ *a.s.* which is well defined in $[-\infty, +\infty)$.

Lemma 2.1 *Suppose* **A1** *is satisfied. If $\gamma<0$ the APARCH-X equation* (1.2) *(or equivalently* (2.1)*) admits a unique strictly stationary, non anticipative and ergodic solution given by* $\boldsymbol{Y}_t = \boldsymbol{B}_{0t} + \sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}\boldsymbol{C}_{0,t-i-1}\right)\boldsymbol{B}_{0,t-k}$. *When $\gamma \geq 0$ there exists no stationary solution to* (1.2) *and to* (2.1).

2.2 Strong consistency of the QMLE

Hamadeh and Zakoian (2011) showed that, for APARCH models, the power parameter δ is difficult to be estimated in practice. We therefore consider that δ is fixed. A generic element of the parameter space $\Theta \subseteq (0,+\infty)\times[0,+\infty)^{d-1}$, where $d = 2q+p+r+1$, is denoted by $\boldsymbol{\vartheta} = (\omega, \alpha_{1+}, \alpha_{1-}, \ldots, \alpha_{q+}, \alpha_{q-}, \beta_1, \ldots, \beta_p, \boldsymbol{\pi}')'$.

Let $(\varepsilon_1, \ldots, \varepsilon_n)$ be a realization of length n of the stationary solution (ε_t) to the APARCH-X model (1.2), and let $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ be the corresponding observations of the exogenous variables. Given initial values $\varepsilon_{1-q}, \ldots, \varepsilon_0$, $\widetilde{\sigma}_{1-p} \geq 0, \ldots, \widetilde{\sigma}_0 \geq 0$, $\boldsymbol{x}_0 \geq 0$, the Gaussian quasi-likelihood is given by

$$L_n(\boldsymbol{\vartheta}) = L_n(\boldsymbol{\vartheta}, \varepsilon_1, \ldots, \varepsilon_n, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi\widetilde{\sigma}_t^2}} \exp\left\{\frac{-\varepsilon_t^2}{2\widetilde{\sigma}_t^2}\right\}$$

where the $\widetilde{\sigma}_t$ are defined recursively, for $t \geq 1$, by

$$\widetilde{\sigma}_t^\delta = \widetilde{\sigma}_t^\delta(\boldsymbol{\vartheta}) = \omega + \sum_{i=1}^{q} \alpha_{i+}\left(\varepsilon_{t-i}^+\right)^\delta + \alpha_{i-}\left(\varepsilon_{t-i}^-\right)^\delta + \sum_{j=1}^{p}\beta_j\widetilde{\sigma}_{t-j}^\delta + \boldsymbol{\pi}'\boldsymbol{x}_{t-1}.$$

The QMLE of $\boldsymbol{\vartheta}_0$ is defined as an any measurable solution $\widehat{\boldsymbol{\vartheta}}_n$ of

$$\widehat{\boldsymbol{\vartheta}}_n = \arg\max_{\boldsymbol{\vartheta}\in\Theta} L_n(\boldsymbol{\vartheta}) = \arg\min_{\boldsymbol{\vartheta}\in\Theta} \widetilde{Q}_n(\boldsymbol{\vartheta}), \tag{2.2}$$

where

$$\widetilde{Q}_n(\boldsymbol{\vartheta}) = \frac{1}{n}\sum_{t=1}^{n} \widetilde{\ell}_t, \qquad \widetilde{\ell}_t = \widetilde{\ell}_t(\boldsymbol{\vartheta}) = \frac{\varepsilon_t^2}{\widetilde{\sigma}_t^2} + \ln \widetilde{\sigma}_t^2. \tag{2.3}$$

Let $\mathcal{A}_{\boldsymbol{\vartheta}+}(z) = \sum_{i=1}^{q} \alpha_{i+} z^i$, $\mathcal{A}_{\boldsymbol{\vartheta}-}(z) = \sum_{i=1}^{q} \alpha_{i-} z^i$ and $\mathcal{B}_{\boldsymbol{\vartheta}}(z) = 1 - \sum_{j=1}^{p} \beta_j z^j$. To show the strong consistency of the QMLE, we need the following assumptions.

A2: $E(\eta_t \mid \mathcal{F}_{t-1}) = 0$ and $E(\eta_t^2 \mid \mathcal{F}_{t-1}) = 1$, where $\mathcal{F}_{t-1}$ denotes the σ-field generated by $\{\varepsilon_u, \boldsymbol{x}_u, u < t\}$.

A3: $\boldsymbol{\vartheta}_0 \in \Theta$, Θ is compact.

A4: for all $i \geq 1$, the support of the distribution of η_{t-i} given $\mathcal{F}_{t,i}$, where $\mathcal{F}_{t,i}$ is a $\sigma-$field generated by $\{\eta_{t-j}, j > i, \boldsymbol{x}_{t-k}, k > 0\}$, is not included in $[0,\infty)$ or in $(-\infty, 0]$ and contains at least three points.

A5: $\gamma < 0$ and $\sum_{j=1}^{p} \beta_j < 1$ for all $\boldsymbol{\vartheta} \in \Theta$.

A6: there exists $s > 0$, such that $Eh_t^s < \infty$ and $E|\varepsilon_t|^s < \infty$.

A7: if $p > 0$, $\mathcal{B}_{\boldsymbol{\vartheta}_0}(z)$ has no common root with $\mathcal{A}_{\boldsymbol{\vartheta}_0+}(z)$ and $\mathcal{A}_{\boldsymbol{\vartheta}_0-}(z)$; $\mathcal{A}_{\boldsymbol{\vartheta}_0+}(1) + \mathcal{A}_{\boldsymbol{\vartheta}_0-}(1) \neq 0$ and $\alpha_{0q+} + \alpha_{0q-} + \beta_{0p} \neq 0$ (with the notation $\alpha_{00+} = \alpha_{00-} = \beta_{00} = 1$).

A8: The variance of $\boldsymbol{x}_1$ is positive definite.

Theorem 2.1 *Let $\widehat{\boldsymbol{\vartheta}}_n$ be a sequence of QMLE satisfying (2.2). Then under* **A1**–**A8***,*

$$\widehat{\boldsymbol{\vartheta}}_n \to \boldsymbol{\vartheta}_0 \ \text{a.s. as } n \to \infty.$$

2.3 Asymptotic distribution of the QMLE

The asymptotic distribution of the estimators depends on the following four cases:

Case A : η_t is independent of $\mathcal{F}_{t-1}$ and all the components of $\boldsymbol{\vartheta}_0$ are strictly positive;

Case B : η_t is independent of $\mathcal{F}_{t-1}$ and at least one component of $\boldsymbol{\vartheta}_0$ is equal to zero;

Case C : η_t is not independent of $\mathcal{F}_{t-1}$ and all the components of $\boldsymbol{\vartheta}_0$ are strictly positive;

Case D : η_t is not independent of $\mathcal{F}_{t-1}$ and at least one component of $\boldsymbol{\vartheta}_0$ is equal to zero.

We assume that

A9: $\mathcal{C} := \lim_{n\to\infty}\sqrt{n}(\Theta - \boldsymbol{\vartheta}_0) = \prod_{i=1}^d \mathcal{C}_i$, where $\mathcal{C}_i = [0, +\infty)$ when $\boldsymbol{\vartheta}_{0i} = 0$ and $\mathcal{C}_i = \mathbb{R}$ otherwise.

A10: $E\eta_t^4 < \infty$ in Cases A and B, and $E|\eta_t|^{4+\nu} < \infty$ for some $\nu > 0$ in Cases C and D.

A11: $E|\varepsilon_t|^{2\delta} < \infty$ and $E\|\boldsymbol{x}_t\|^2 < \infty$ in Case B, and $E|\varepsilon_t|^{2\delta+8\delta/\nu} < \infty$ and $E\|\boldsymbol{x}_t\|^{2+8/\nu} < \infty$ in Case D.

A12: in Cases B and D, there exist Hölder conjugate numbers p and $q > 1$ such that

$$p^{-1} + q^{-1} = 1 \quad \text{and} \quad E|\varepsilon_t|^{2\delta q} < \infty, \quad E|\varepsilon_t|^{2p} < \infty, \quad E\|\boldsymbol{x}_t\|^{2q} < \infty.$$

Theorem 2.2 *Under the assumptions of Theorem 2.1 and* **A9**–**A12***, as* $n \to \infty$*,*

$$\sqrt{n}(\widehat{\boldsymbol{\vartheta}}_n - \boldsymbol{\vartheta}_0) \xrightarrow{d} \boldsymbol{Z}^{\mathcal{C}}, \text{ where } \boldsymbol{Z} \sim \mathcal{N}\left\{0, \boldsymbol{J}^{-1}\boldsymbol{I}\boldsymbol{J}^{-1}\right\}, \tag{2.4}$$

$$\boldsymbol{J} := E\left(\frac{\partial^2 \ell_t(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\vartheta}\partial\boldsymbol{\vartheta}'}\right) = \frac{4}{\delta^2}E\left(\frac{1}{\sigma_t^{2\delta}(\boldsymbol{\vartheta}_0)}\frac{\partial\sigma_t^\delta(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\vartheta}}\frac{\partial\sigma_t^\delta(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\vartheta}'}\right) \tag{2.5}$$

and

$$\boldsymbol{I} = \frac{4}{\delta^2}E\left[\left\{E\left(\eta_t^4 \mid \mathcal{F}_{t-1}\right) - 1\right\}\frac{1}{\sigma_t^{2\delta}(\boldsymbol{\vartheta}_0)}\frac{\partial\sigma_t^\delta(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\vartheta}}\frac{\partial\sigma_t^\delta(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\vartheta}'}\right]$$

and $\boldsymbol{Z}^{\mathcal{C}} = \arg\inf_{\boldsymbol{C}\in\mathcal{C}}\|\boldsymbol{C} - \boldsymbol{Z}\|_{\boldsymbol{J}}$.

The next proposition provides estimates for the matrices $\boldsymbol{I}$ and $\boldsymbol{J}$ required to apply Theorem 2.2. Assumption **A12** needs to be slightly reinforced as follow.

A12': in Cases B and D, there exist Hölder conjugate numbers p and $q > 1$ such that

$$p^{-1} + q^{-1} = 1 \quad \text{and} \quad E|\varepsilon_t|^{2\delta q} < \infty, \quad E|\varepsilon_t|^{4p} < \infty, \quad E\|\boldsymbol{x}_t\|^{2q} < \infty.$$

Proposition 2.1 *Under the assumptions of Theorem 2.2 with* **A12** *replaced by* **A12'***, strongly consistent estimators of* $\boldsymbol{J}$ *and* $\boldsymbol{I}$ *are given by*

$$\widehat{\boldsymbol{J}}_n = \frac{4}{\delta^2}\frac{1}{n}\sum_{t=1}^n \frac{1}{\widetilde{\sigma}_t^{2\delta}(\widehat{\boldsymbol{\vartheta}}_n)}\frac{\partial\widetilde{\sigma}_t^\delta(\widehat{\boldsymbol{\vartheta}}_n)}{\partial\boldsymbol{\vartheta}}\frac{\partial\widetilde{\sigma}_t^\delta(\widehat{\boldsymbol{\vartheta}}_n)}{\partial\boldsymbol{\vartheta}'} \tag{2.6}$$

and

$$\widehat{\boldsymbol{I}}_n = \frac{4}{\delta^2}\frac{1}{n}\sum_{t=1}^n \left(\widehat{\eta}_t^4 - 1\right)\frac{1}{\widetilde{\sigma}_t^{2\delta}(\widehat{\boldsymbol{\vartheta}}_n)}\frac{\partial\widetilde{\sigma}_t^\delta(\widehat{\boldsymbol{\vartheta}}_n)}{\partial\boldsymbol{\vartheta}}\frac{\partial\widetilde{\sigma}_t^\delta(\widehat{\boldsymbol{\vartheta}}_n)}{\partial\boldsymbol{\vartheta}'}, \tag{2.7}$$

with $\widehat{\eta}_t = \varepsilon_t/\widetilde{\sigma}_t(\widehat{\boldsymbol{\vartheta}}_n)$.

Remark 2.1 *In Cases A and B, the estimator defined by* (2.7) *can be replaced by*

$$\widehat{\boldsymbol{I}}_n = \left(\frac{1}{n} \sum_{t=1}^{n} \widehat{\eta}_t^4 - 1 \right) \widehat{\boldsymbol{J}}_n. \tag{2.8}$$

Let $\boldsymbol{e}_k$ be the k-th element of the canonical basis of $\mathbb{R}^d$. We will test the hypothesis that the k-th element of $\boldsymbol{\vartheta}_0$ is equal to zero, assuming that the other elements are positive:

$$H_0 : \boldsymbol{e}_k'\boldsymbol{\vartheta}_0 = 0 \text{ and } \boldsymbol{e}_\ell'\boldsymbol{\vartheta}_0 > 0 \quad \forall \ell \neq k \qquad \text{against} \qquad H_1 : \boldsymbol{e}_k'\boldsymbol{\vartheta}_0 > 0. \tag{2.9}$$

For this testing problem, the Student t-test statistic is defined by

$$t_n(k) = \frac{\boldsymbol{e}_k'\widehat{\boldsymbol{\vartheta}}_n}{\sqrt{\boldsymbol{e}_k'\widehat{\boldsymbol{\Sigma}}\boldsymbol{e}_k}}, \qquad \widehat{\boldsymbol{\Sigma}} = \widehat{\boldsymbol{J}}_n^{-1}\widehat{\boldsymbol{I}}_n\widehat{\boldsymbol{J}}_n^{-1}.$$

Denote by $\chi^2_\ell(\alpha)$ the α-quantile of the chi-squared distribution with ℓ degrees of freedom. As a corollary of Theorem 2.2 and Proposition 2.1, we obtain the following result.

Corollary 2.1 *Under the assumptions of Theorem 2.2, the test of rejection region*

$$\{t_n^2(k) > \chi^2_1(1 - 2\alpha)\}$$

has the asymptotic level α under H_0 and is consistent under H_1 defined in (2.9).

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